Graph Pooling: DIFFPOOL

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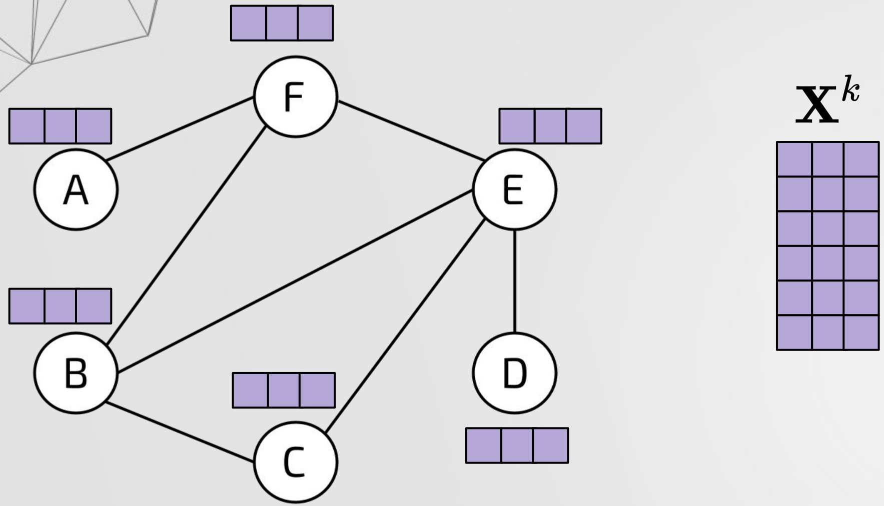
TIM[2]
EIT DIGITAL[3]

01 Graph prediction



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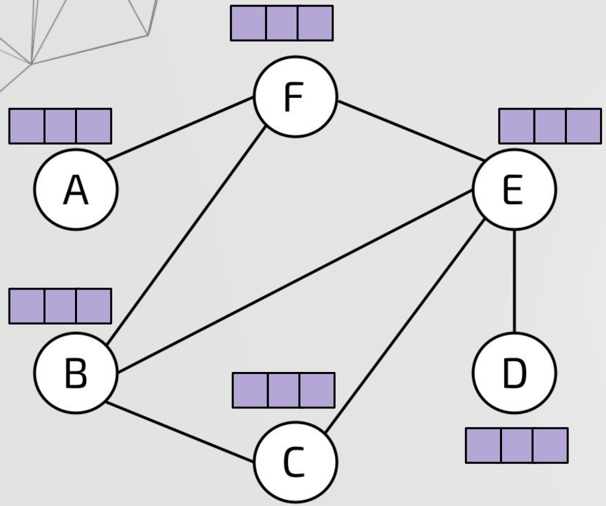

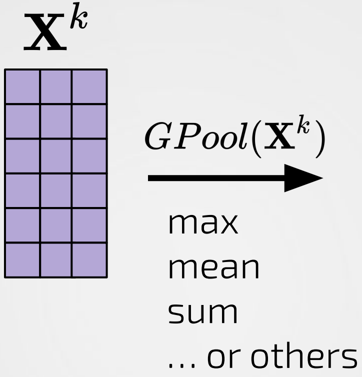

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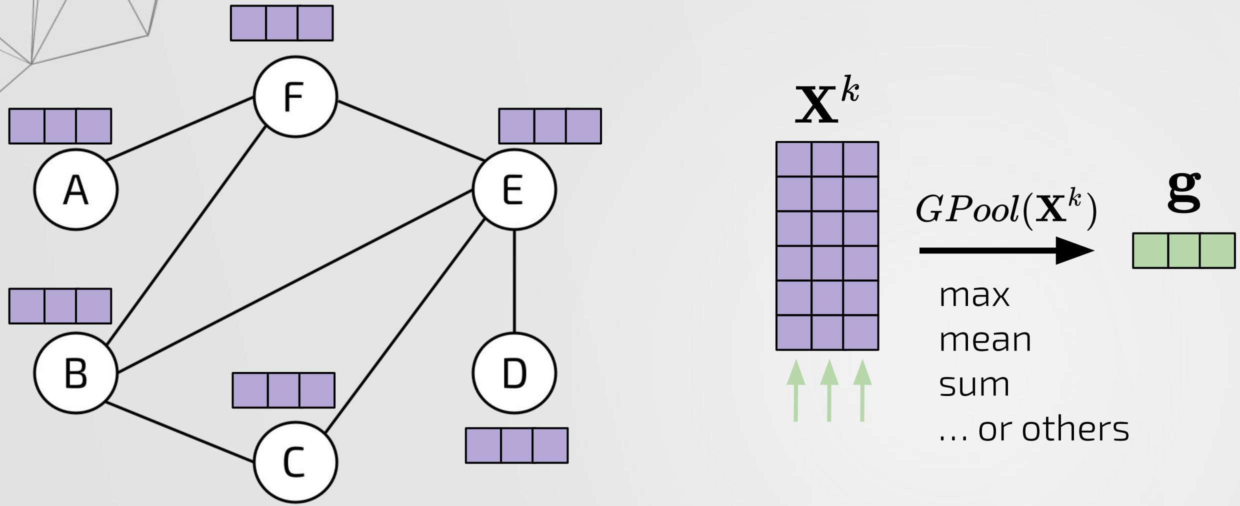

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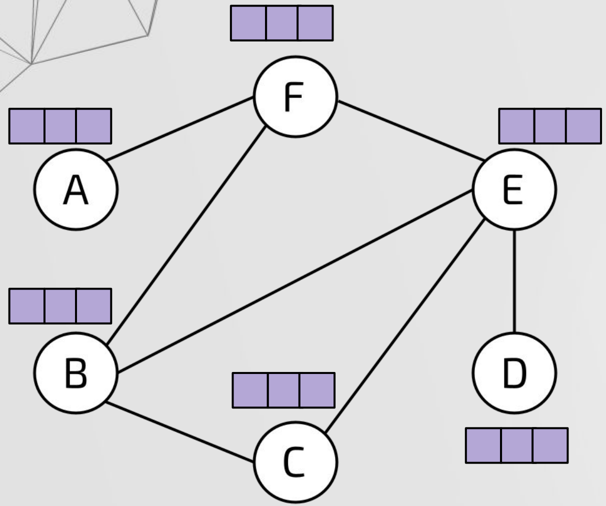

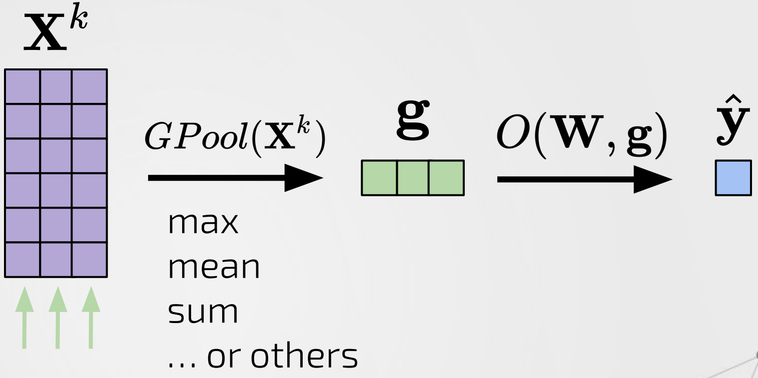

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$O(\mathbf{W}, \mathbf{g}) \longrightarrow$ graph readout function

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Pooling the node embeddings all together do not capture hierarchical representations of the network

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DIFFPOOL: hierarchical nodes pooling strategy

02 DIFFPOOL[1]

DIFFerentiable **POOL**ing: Compute an hierarchical representation of the graph by aggregating "close" nodes

Ying et al, *Hierarchical Graph Representation Learning with Differentiable Pooling,* NIPS '18

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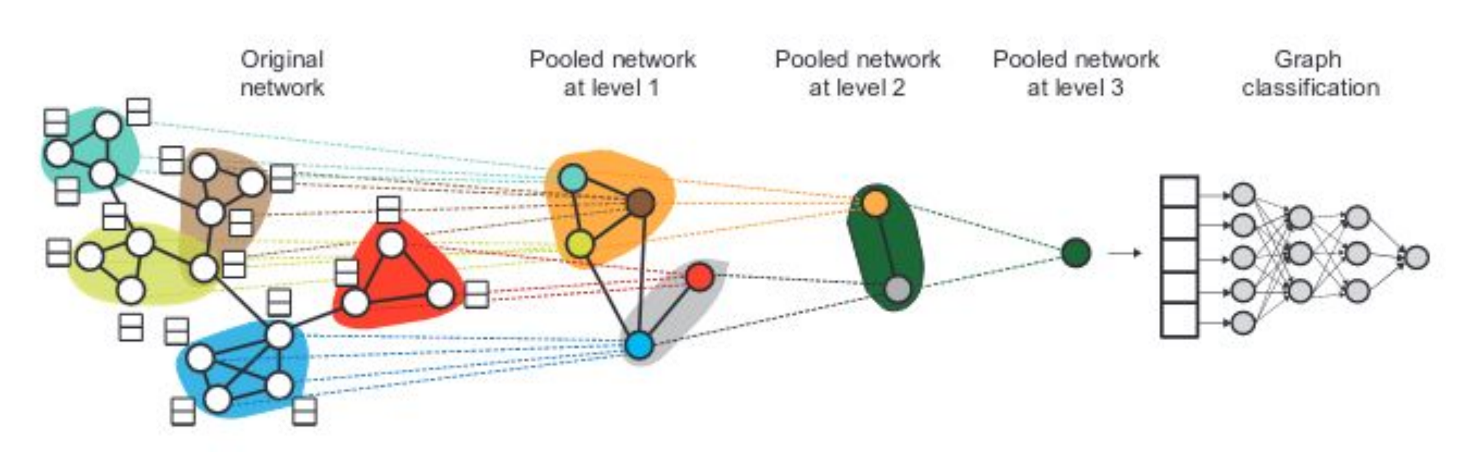
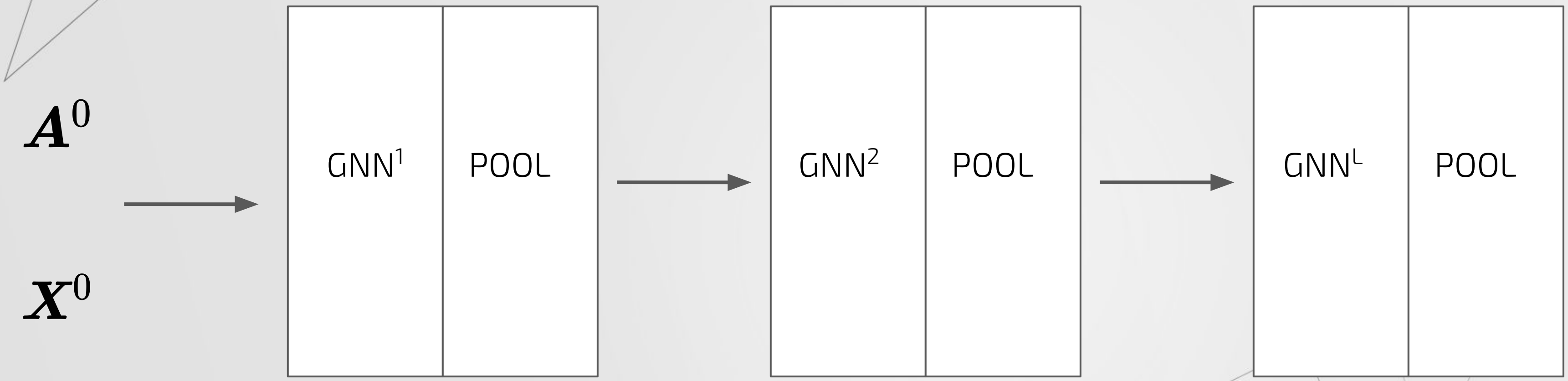


Image taken from the original publication.

Ying et al, *Hierarchical Graph Representation Learning with Differentiable Pooling,* NIPS ‘18

02 DIFFPOOL

Idea: stack several GNN and pooling layer on top of each other



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adjacency matrix

node features matrix

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arbitrary GNN that computes K iterations, $Z = X^K$

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with $m < n$

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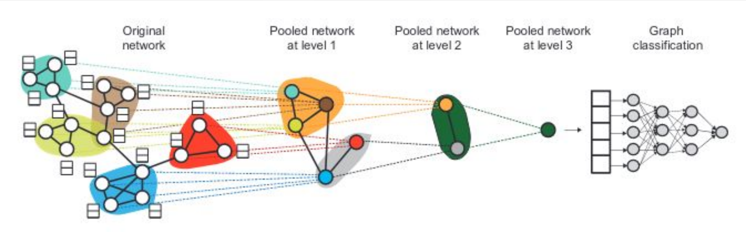
Solution -> learn a cluster assignment for each node

03 Pooling in DIFFPOOL

The final representation is obtained by coarsening the graph in L hierarchical steps. To do that, at each step a **cluster assignment matrix** is learned.



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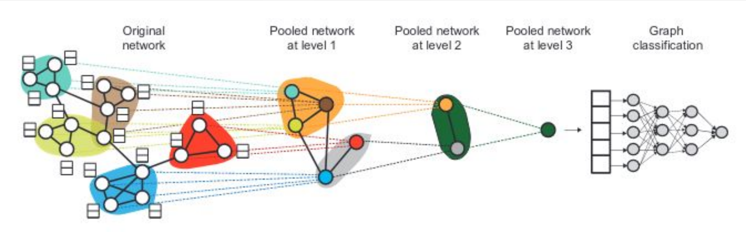

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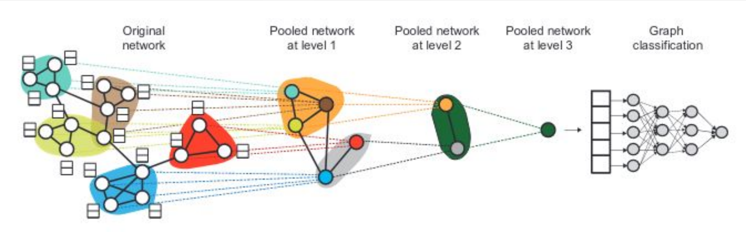

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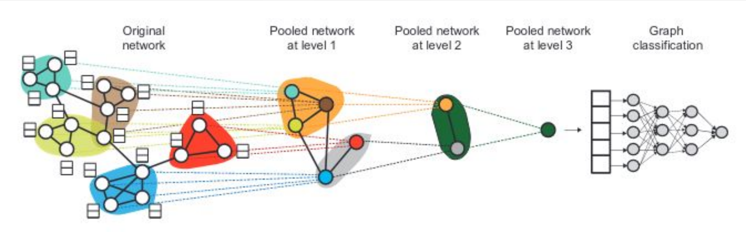

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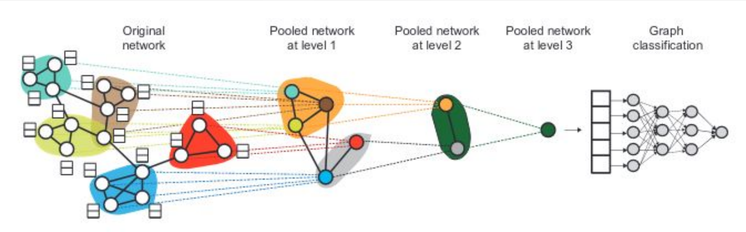

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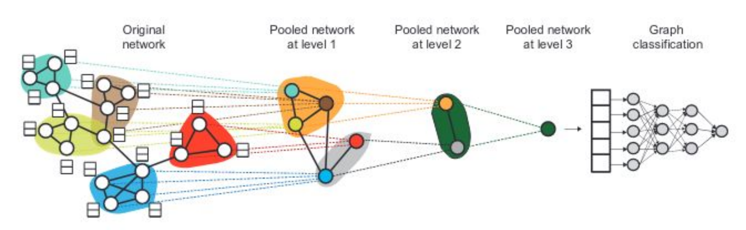

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The number of clusters at each step is an hyperparameter!

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differentiable!

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At the last step the graph is condensed in one single node (vector). The final prediction is the output of a MLP.

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Let's switch to the notebook...