Chapter 5

Other Parametric Distributions

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Based on the authors' text *Statistical Methods for Reliability Data*, John Wiley & Sons Inc. 1998.

July 18, 2002
12h 24min

Other Parametric Distributions
Chapter 5 Objectives

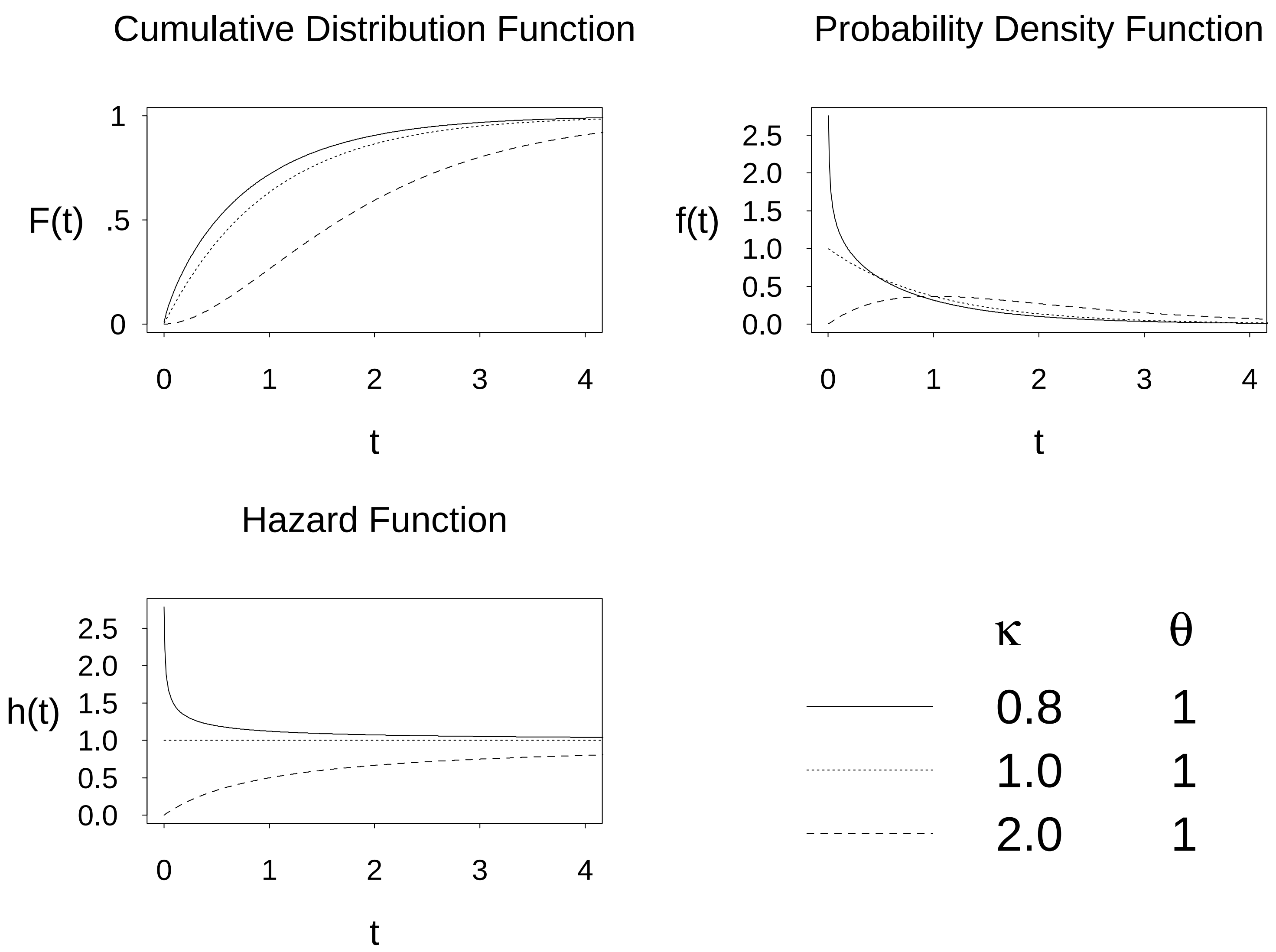
- Describe the properties and the importance of the following parametric distributions which cannot be transformed into a location-scale distribution:

  Gamma, Generalized Gamma, Extended Generalized Gamma, Generalized F, Inverse Gaussian, Birnbaum–Saunders, Gompertz–Makeham.

- Introduce the concept of a threshold-parameter distribution.

- Illustrate how other statistical models can be determined by applying basic ideas of probability theory to physical properties of a failure process, system, or population of units.

Examples of Gamma Distributions



Gamma Distribution

- T follows a gamma distribution, GAM(θ, κ), if

$$\begin{aligned} F(t;\theta,\kappa) &= \Gamma_{\mathrm{I}}\left(\frac{t}{\theta};\kappa\right) \\ f(t;\theta,\kappa) &= \frac{1}{\Gamma(\kappa)\,\theta}\left(\frac{t}{\theta}\right)^{\kappa-1}\exp\left(-\frac{t}{\theta}\right), \quad t>0 \end{aligned}$$

  $\theta > 0$ is a scale parameter and $\kappa > 0$ is a shape parameter. $\Gamma_{\mathrm{I}}(v;\kappa)$ is the incomplete gamma function defined by

$$\Gamma_{\mathrm{I}}(v;\kappa) = \frac{\int_0^v x^{\kappa-1}\exp(-x)\,dx}{\Gamma(\kappa)}, \quad v \geq 0.$$

- **Special case:** when $\kappa = 1$, GAM$(\theta,\kappa) \equiv$ EXP(θ).

- The hazard function $h(t;\theta,\kappa)$ is **decreasing** when $\kappa < 1$; **increasing** when $\kappa > 1$; and **approaches a constant** level late in life i.e.,

$$\lim_{t\to\infty} h(t;\theta,\kappa) = 1/\theta.$$

Moments and Quantiles of the Gamma Distribution

- **Moments:** For integer $m > 0$

$$\mathsf{E}(T^m) = \frac{\theta^m \, \Gamma\,(m + \kappa)}{\Gamma(\kappa)}.$$

Then

$$\begin{aligned} \mathsf{E}(T) &= \theta\kappa \\ \mathsf{Var}(T) &= \theta^2\kappa \end{aligned}$$

- **Quantiles:** the p quantile of the distribution is given by

$$t_p = \theta \, \Gamma_{\mathrm{I}}^{-1}(p; \kappa).$$

Reparameterization of the Gamma Distribution

For accelerated time regression modeling, the cdf and pdf can be conveniently **reparameterized** as follows:

$$
\begin{aligned}
F(t;\theta,\kappa) &= \Phi_{\mathrm{lg}}\left[\log(t)-\mu;\kappa\right] \\
f(t;\theta,\kappa) &= \frac{1}{t}\,\phi_{\mathrm{lg}}\left[\log(t)-\mu;\kappa\right]
\end{aligned}
$$

where $\mu = \log(\theta)$, Φ_{lg} and ϕ_{lg} are the cdf and pdf for the **standardized** loggamma variable $Z = \log(T/\theta) = \log(T) - \mu$,

$$
\begin{aligned}
\Phi_{\mathrm{lg}}(z;\kappa) &= \Gamma_{\mathrm{I}}[\exp(z);\kappa] \\
\phi_{\mathrm{lg}}(z;\kappa) &= \frac{1}{\Gamma(\kappa)}\exp\left[\kappa z-\exp(z)\right].
\end{aligned}
$$

Generalized Gamma Distribution

- T has a generalized gamma distribution if

$$F(t;\theta,\beta,\kappa) = \Gamma_{\mathrm{I}}\left[\left(\frac{t}{\theta}\right)^{\beta};\kappa\right]$$

$$f(t;\theta,\beta,\kappa) = \frac{\beta}{\Gamma(\kappa)\theta}\left(\frac{t}{\theta}\right)^{\kappa\beta-1}\exp\left[-\left(\frac{t}{\theta}\right)^{\beta}\right], \quad t > 0$$

  where $\theta > 0$ is a scale parameter, and $\kappa > 0$, $\beta > 0$ are shape parameters.

- If $\beta = 1$ the distribution becomes the GAM(θ, κ) distribution.

- If $\kappa = 1$ the distribution becomes the WEIB(μ, σ), where $\mu = \log(\theta)$ and $\sigma = 1/\beta$.

- If $\beta = 1$ and $\kappa = 1$ the distribution becomes the EXP(θ) distribution.

Generalized Gamma Distribution-Continued

- A more convenient parameterization is given by $\mu = \log(\theta) + (\sigma/\lambda)\log(\lambda^{-2})$, $\lambda = 1/\sqrt{\kappa}$, and $\sigma = 1/(\beta\sqrt{\kappa})$, in which case, we write $T \sim \text{GENG}(\mu, \sigma, \lambda)$ and

$$\begin{aligned} F(t;\mu,\sigma,\lambda) &= \Phi_{\text{lg}}\left[\lambda\omega + \log(\lambda^{-2}); \lambda^{-2}\right] \\ f(t;\mu,\sigma,\lambda) &= \frac{\lambda}{\sigma t}\phi_{\text{lg}}\left[\lambda\omega + \log(\lambda^{-2}); \lambda^{-2}\right] \end{aligned}$$

where $\omega = [\log(t) - \mu]/\sigma$, $-\infty < \mu < \infty$, $\sigma > 0$, and $\lambda > 0$.

- If $T \sim \text{GENG}(\mu, \sigma, \lambda)$ and $c > 0$ then $cT \sim \text{GENG}[\mu - \log(c), \lambda, \sigma]$.

- As $\lambda \to 0$, $T \dot\sim \text{LOGNOR}(\mu, \sigma)$.

- Moments, quantiles, and other related distributions will follow as special cases of the more general extended generalized gamma distribution.

Extended Generalized Gamma Distribution

- T has an extended generalized gamma distribution, EGENG(μ, σ, λ), if

$$F(t;\mu,\sigma,\lambda) = \begin{cases} \Phi_{\text{lg}}\left[\lambda\omega + \log(\lambda^{-2}); \lambda^{-2}\right] & \text{if } \lambda > 0 \\ \Phi_{\text{nor}}(\omega) & \text{if } \lambda = 0 \\ 1 - \Phi_{\text{lg}}\left[\lambda\omega + \log(\lambda^{-2}); \lambda^{-2}\right] & \text{if } \lambda < 0 \end{cases}$$

$$f(t;\mu,\sigma,\lambda) = \begin{cases} \frac{|\lambda|}{\sigma t}\phi_{\text{lg}}\left[\lambda\omega + \log(\lambda^{-2}); \lambda^{-2}\right] & \text{if } \lambda \neq 0 \\ \frac{1}{\sigma t}\phi_{\text{nor}}(\omega) & \text{if } \lambda = 0 \end{cases}$$

where $\omega = [\log(t) - \mu]/\sigma$, $-\infty < \mu < \infty$, $\exp(\mu)$ is a scale parameter, $-\infty < \lambda < \infty$ and $\sigma > 0$ are shape parameters.

Comments on the EGENG Distribution

- The distribution at $\lambda = 0$ is defined by **continuity** (i.e., the limiting distribution when $\lambda \to 0$).

- If $T \sim \text{EGENG}(\mu, \sigma, \lambda)$ and $c > 0$ then $cT \sim \text{EGENG}[\mu - \log(c), \lambda, \sigma]$. Thus, $\exp(\mu)$ is a location-parameter for T.

- When $T \sim \text{EGENG}(\mu, \lambda, \sigma)$ then the distribution of $W = [\log(T) - \mu]/\sigma$ depends only on λ.

- Note that for each fixed λ, $\log(T)$ is location-scale (μ, σ) with a standardized location-scale distribution equal to the distribution of W.

Extended Generalized Gamma Distribution–Continued

- **Moments:** For integer m and $\lambda \neq 0$

$$\mathsf{E}(T^m) = \begin{cases} \dfrac{\exp(m\mu)\left(\lambda^2\right)^{m\sigma/\lambda}\Gamma\left[\lambda^{-1}(m\sigma+\lambda^{-1})\right]}{\Gamma(\lambda^{-2})} & \text{if } m\lambda\sigma + 1 > 0 \\ \infty & \text{if } m\lambda\sigma + 1 \leq 0. \end{cases}$$

When $\lambda = 0$, the moments are

$$\mathsf{E}(T^m) = \exp\left[m\mu + (1/2)(m\sigma)^2\right].$$

- Thus when the mean and the variance are finite and $\lambda \neq 0$,

$$\mathsf{E}(T) = \frac{\theta\,\Gamma\left[\lambda^{-1}(\sigma + \lambda^{-1})\right]}{\Gamma(\lambda^{-2})}$$

$$\mathsf{Var}(T) = \theta^2 \left[\frac{\Gamma\left[\lambda^{-1}(2\sigma + \lambda^{-1})\right]}{\Gamma(\lambda^{-2})} - \frac{\Gamma^2\left[\lambda^{-1}(\sigma + \lambda^{-1})\right]}{\Gamma^2(\lambda^{-2})}\right].$$

- When $\lambda = 0$, $\mathsf{E}(T) = \exp[\mu + (1/2)\sigma^2]$ and $\mathsf{Var}(T) = \exp(2\mu + \sigma^2) \times [\exp(\sigma^2) - 1]$.

Quantiles of the EGENG Distribution

The EGENG quantiles are

$$\log(t_p) = \mu + \sigma \times \omega(p;\lambda)$$

where $\omega(p;\lambda)$ is the p quantile of the distribution of W,

$$\omega(p;\lambda) = \begin{cases} \lambda^{-1}\log\left[\lambda^2\Gamma_{\mathrm{I}}^{-1}(p;\lambda^{-2})\right] & \text{if } \lambda > 0 \\ \Phi_{\text{nor}}^{-1}(p) & \text{if } \lambda = 0 \\ \lambda^{-1}\log\left[\lambda^2\Gamma_{\mathrm{I}}^{-1}(1-p;\lambda^{-2})\right] & \text{if } \lambda < 0 \end{cases}$$

Distributions Related to EGENG

Special Cases:

- If $\lambda > 0$ then EGENG(μ, σ, λ) = GENG(μ, σ, λ).
- if $\lambda = 1$, $T \sim$ WEIB(μ, σ).
- if $\lambda = 0$, $T \sim$ LOGNOR(μ, σ).
- if $\lambda = -1$, $1/T \sim$ WEIB$(-\mu, \sigma)$, [i.e., T has a reciprocal Weibull (or Fréchet distribution of maxima)].
- When $\lambda = \sigma$, $T \sim$ GAM(θ, κ), where $\theta = \lambda^2 \exp(\mu)$ and $\kappa = \lambda^{-2}$.
- When $\lambda = \sigma = 1$, $T \sim$ EXP(θ), where $\theta = \lambda^2 \exp(\mu)$.

Comment on EGENG(μ, σ, λ) Parameterization

- The (μ, σ, λ) parameterization is due to Farewell and Prentice (1977). Observe that

$$F[\exp(\mu); \mu, \sigma, \lambda] = \begin{cases} \Gamma_{\mathrm{I}}(\lambda^{-2}; \lambda^{-2}) & \text{if } \lambda > 0 \\ .5 & \text{if } \lambda = 0 \\ 1 - \Gamma_{\mathrm{I}}(\lambda^{-2}; \lambda^{-2}) & \text{if } \lambda < 0 \end{cases}$$

  This value of $F[\exp(\mu); \mu, \sigma, \lambda]$, as a function of λ, is always in the interval $[.5, 1)$. Thus $\exp(\mu)$ equals a quantile t_p with $p \geq .5$.

- The parameterization is stable when there is not much censoring. It tends to be unstable when there is heavy censoring.

- When there is heavy censoring a different parameterization is needed for ML estimation.

EGENG **Stable Parameterization**

- **Parameterization for Numerical Stability:** with $p_1 < p_2$, an stable parameterization can be obtained using two quantiles (t_{p_1}, t_{p_2}), and λ, i.e.,

$$\begin{aligned} \log(t_{p_1}) &= \mu + \sigma\omega(p_1, \lambda) \\ \log(t_{p_2}) &= \mu + \sigma\omega(p_2, \lambda) \end{aligned}$$

and solving for μ and σ,

$$\begin{aligned} \mu &= \frac{\omega(p_2, \lambda) \times \log(t_{p_1}) - \omega(p_1, \lambda) \times \log(t_{p_2})}{\omega(p_2, \lambda) - \omega(p_1, \lambda)} \\ \sigma &= \frac{\log(t_{p_2}) - \log(t_{p_1})}{\omega(p_2, \lambda) - \omega(p_1, \lambda)}. \end{aligned}$$

Generalized F Distribution

T has a generalized F distribution with parameters (μ, σ, κ, r), say GENF(μ, σ, κ, r), if

$$F_T(t; \mu, \sigma, \kappa, r) = \Phi_{\mathsf{lf}}\left[\frac{\log(t) - \mu}{\sigma}; \kappa, r\right]$$

$$f_T(t; \mu, \sigma, \kappa, r) = \frac{1}{\sigma t}\,\phi_{\mathsf{lf}}\left[\frac{\log(t) - \mu}{\sigma}; \kappa, r\right], \quad t > 0$$

where

$$\phi_{\mathsf{lf}}(z; \kappa, r) = \frac{\Gamma(\kappa + r)}{\Gamma(\kappa)\,\Gamma(r)}\,\frac{(\kappa/r)^{\kappa}\exp(\kappa z)}{[1 + (\kappa/r)\exp(z)]^{\kappa + r}}$$

is the pdf of the central log F distribution with 2κ and $2r$ degrees of freedom and Φ_{lf} is the corresponding cdf.

It follows that $\phi_{\mathsf{lf}}(z; \kappa, r)$ and $\Phi_{\mathsf{lf}}(z; \kappa, r)$ are the pdf and cdf of $Z = [\log(T) - \mu]/\sigma$.

$\exp(\mu)$ is a scale parameter and $\sigma > 0$, $\kappa > 0$, $r > 0$ are shape parameters.

Generalized F Distribution-Continued

- **Moments:** For integer $m \geq 0$,

$$E(T^m) = \begin{cases} \frac{\exp(m\mu)\,\Gamma(\kappa+m\sigma)\,\Gamma(r-m\sigma)}{\Gamma(\kappa)\,\Gamma(r)} \left(\frac{r}{\kappa}\right)^{m\sigma}, & \text{if } r > m\sigma \\ \infty & \text{otherwise.} \end{cases}$$

Then

$$E(T) = \frac{\Gamma(\kappa+\sigma)\,\Gamma(r-\sigma)}{\Gamma(\kappa)\Gamma(r)} \exp(\mu) \left(\frac{r}{\kappa}\right)^{\sigma}$$

$$\text{Var}(T) = \left\{\frac{\Gamma(\kappa+2\sigma)\,\Gamma(r-2\sigma)}{\Gamma(\kappa)\Gamma(r)} - \frac{\Gamma^2(\kappa+\sigma)\,\Gamma^2(r-\sigma)}{\Gamma^2(\kappa)\Gamma^2(r)}\right\} \exp(2\mu) \left(\frac{r}{\kappa}\right)^{2\sigma}$$

where $r > \sigma$ for the mean and $r > 2\sigma$ for the variance.

- **Quantiles:** The p quantile of the distribution is

$$t_p = \exp(\mu) \left[\mathcal{F}_{(p,2\kappa,2r)}\right]^{\sigma}$$

where $\mathcal{F}_{(p,2\kappa,2r)}$ is the p quantile of an F distribution with $(2\kappa, 2r)$ degrees of freedom.

The expression for t_p follows directly from the fact that $T = \exp(\mu)V^{\sigma}$ where V has an F distribution with $(2\kappa, 2r)$ degrees of freedom.

Generalized F Distribution–Special Cases

- $1/T \sim \text{GENF}(-\mu, \sigma, r, \kappa)$.
- When $(\mu, \sigma) = (0, 1)$ then T follows an F distribution with 2κ numerator and $2r$ denominator degrees of freedom.
- When $(\kappa, r) = (1, 1)$, $\text{GENF}(\mu, \sigma, \kappa, r) \equiv \text{LOGLOGIS}(\mu, \sigma)$.
- When $r \to \infty$, $T \dot\sim \text{GENG}[\exp(\mu)/\kappa^{\sigma}, 1/\sigma, \kappa]$.
- When $(\kappa, r) = (1, \infty)$, $T \sim \text{WEIB}(\mu, \sigma)$.
- When $\kappa = 1$, T follows a Burr type XII distribution with cdf

$$F(t; \mu, \sigma, r) = 1 - \frac{1}{\left[1 + \frac{1}{r}\left(\frac{t}{\theta}\right)^{\frac{1}{\sigma}}\right]^{r}}, \quad t > 0$$

  where $r > 0$, $\sigma > 0$ are shape parameters, and $\theta = \exp(\mu)$ is a scale parameter.
- When $\kappa \to \infty$, and $r \to \infty$, $T \dot\sim \text{LOGNOR}\left(\mu, \sigma\sqrt{(\kappa + r)/\kappa r}\right)$.

Inverse Gaussian Distribution

- A common parameterization for the cdf of this distribution is (see Chhikara and Folks 1989) is

$$\Pr(T \le t; \theta, \lambda) \;=\; \Phi_{\text{nor}}\left[\frac{(t-\theta)\sqrt{\lambda}}{\theta\sqrt{t}}\right] + \exp\left(\frac{2\lambda}{\theta}\right)\Phi_{\text{nor}}\left[-\frac{(t+\theta)\sqrt{\lambda}}{\theta\sqrt{t}}\right],$$

  $t > 0$; $\theta > 0$ and $\lambda > 0$ are parameters in the same units of T.

- Wald (1947) derived this distribution as a limiting form for the distribution of sample size in sequential probability ratio test.

Inverse Gaussian Distribution–Origin

- The inverse Gaussian distribution was originally given by Schrödinger (1915) as the distribution of the first passage time in Brownian motion. The parameters θ and λ relate to the Brownian motion parameters as follows:

- Consider a Brownian process

$$B(t) = ct + dW(t), \quad t > 0$$

where c, d are constants and $W(t)$ is a Wiener process. Let T be the first passage time of a specified level b_0, say

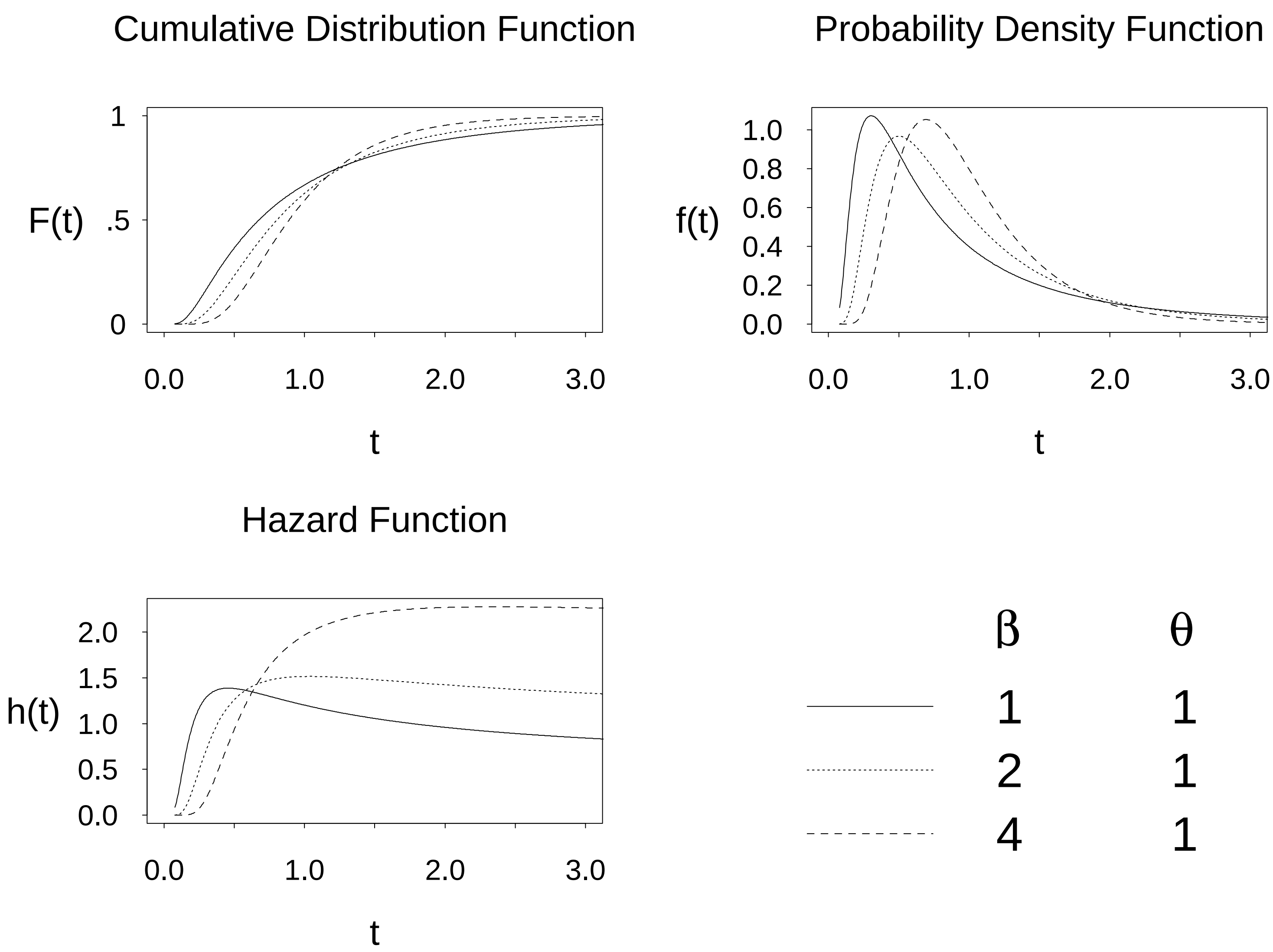
$$T = \inf\left\{t; B(t) \geq b_0\right\}.$$

Then

$$\Pr(T \leq t) = \Phi_{\text{nor}}\left[\frac{(t-\theta)\sqrt{\lambda}}{\theta\sqrt{t}}\right] + \exp\left(\frac{2\lambda}{\theta}\right)\Phi_{\text{nor}}\left[-\frac{(t+\theta)\sqrt{\lambda}}{\theta\sqrt{t}}\right]$$

where $\theta = b_0/c$ and $\sqrt{\lambda} = b_0/d$. Tweedie (1945) gives more details on this approach.

Examples of Inverse Gaussian Distributions



Inverse Gaussian Distribution–Continued

- The reparameterization $(\theta, \beta = \lambda/\theta)$ separates the location and scale parameters. We say that $T \sim \text{IGAU}(\theta, \beta)$ if

$$
\begin{aligned}
F_T(t;\theta,\beta) &= \Phi_{\text{ligau}}\left[\log(t/\theta);\beta\right] \\
f_T(t;\theta,\beta) &= \frac{1}{t}\,\phi_{\text{ligau}}\left[\log(t/\theta);\beta\right], \quad t > 0
\end{aligned}
$$

where $\theta > 0$ is a scale parameter, $\beta > 0$ is at unit less shape parameter, and

$$
\begin{aligned}
\Phi_{\text{ligau}}(z;\beta) &= \Phi_{\text{nor}}\left\{\sqrt{\beta}\left[\frac{\exp(z)-1}{\exp(z/2)}\right]\right\} + \\
&\quad \exp(2\beta)\,\Phi_{\text{nor}}\left\{-\sqrt{\beta}\left[\frac{\exp(z)+1}{\exp(z/2)}\right]\right\} \\
\phi_{\text{ligau}}(z;\beta) &= \frac{\sqrt{\beta}}{\exp(z/2)}\,\phi_{\text{nor}}\left\{\sqrt{\beta}\left[\frac{\exp(z)-1}{\exp(z/2)}\right]\right\}, \quad -\infty < z < \infty.
\end{aligned}
$$

- The hazard function has the following behavior: $h_T(0;\theta,\beta) = 0$, $h_T(t;\theta,\beta)$ is unimodal, and $\lim_{t\to\infty} h_T(t;\theta,\beta) = \beta/(2\theta)$.

Inverse Gaussian Distribution-Continued

- **Moments:** For integer $m > 0$

$$\mathsf{E}(T^m) = \theta^m \sum_{i=0}^{m-1} \frac{(m-1+i)!}{i!\,(m-1-i)!} \left(\frac{1}{2\beta}\right)^i.$$

From this it follows that

$$\mathsf{E}(T) = \theta \quad \text{and} \quad \mathsf{Var}(T) = \theta^2/\beta.$$

- **Quantiles:** the p quantile of the IGAU distribution is

$$t_p = \theta\,\Phi^{-1}_{\mathsf{ligau}}(p;\beta).$$

There is no simple closed form equation for $\Phi^{-1}_{\mathsf{ligau}}(p;\beta)$, so it must be computed by inverting $p = \Phi_{\mathsf{ligau}}(z;\beta)$ numerically.

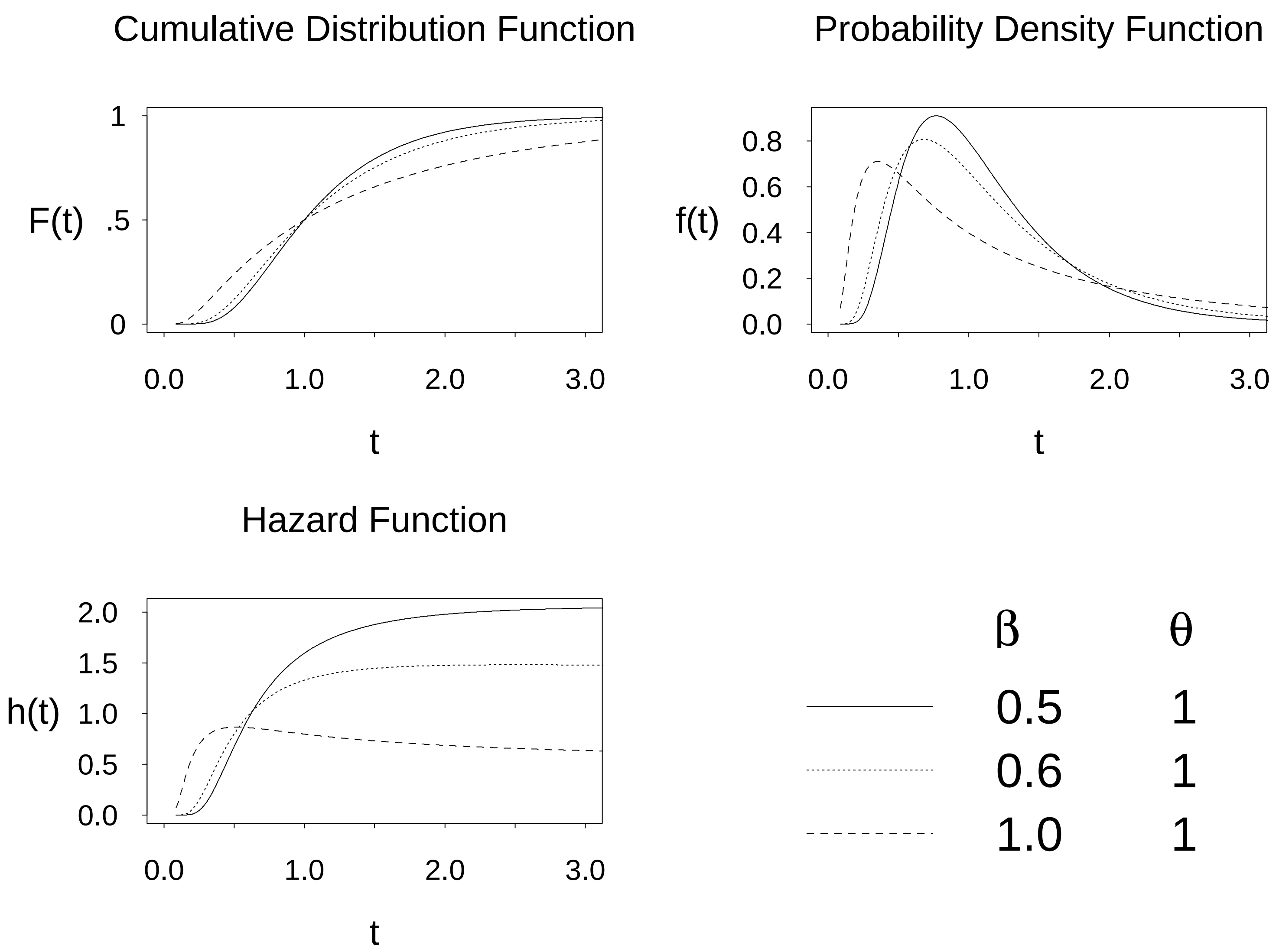
Inverse Gaussian Distribution–Continued

Special cases:

- If $T \sim \text{IGAU}(\theta, \beta)$ and $c > 0$ then $cT \sim \text{IGAU}(c\theta, \beta)$.

- For large values of β, the distribution is very similar to a $\text{NOR}(\theta, \theta/\sqrt{\beta})$.

Examples of Birnbaum–Saunders Distributions



Birnbaum–Saunders Distribution

- For a variable T with Birnbaum–Saunders distribution, BISA(θ, β),
$$\begin{aligned} F_T(t;\beta,\theta) &= \Phi_{\text{nor}}(\zeta) \\ f_T(t;\beta,\theta) &= \frac{\sqrt{\frac{t}{\theta}} + \sqrt{\frac{\theta}{t}}}{2\beta t}\,\phi_{\text{nor}}(\zeta) \end{aligned}$$
where $t \geq 0$, $\theta > 0$ is a scale parameter, $\beta > 0$ is a shape parameter, and
$$\zeta = \frac{1}{\beta}\left(\sqrt{\frac{t}{\theta}} - \sqrt{\frac{\theta}{t}}\right)$$

- **Moments:** For an integer $m > 0$,
$$\mathrm{E}(T^m) = \theta^m \sum_{i=0}^{m} \beta^{2(m-i)} \frac{[2(m-i)]!}{\left[2^{3(m-i)}\right]\,(m-i)!} \sum_{k=0}^{m-i} \binom{2m}{2k} \binom{m-k}{i}.$$
Then
$$\mathrm{E}(T) = \theta\left(1 + \frac{\beta^2}{2}\right) \quad \text{and} \quad \mathrm{Var}(T) = (\theta\beta)^2\left(1 + \frac{5\beta^2}{4}\right).$$

- **Quantiles:** The p quantile is
$$t_p = \frac{\theta}{4}\left\{\beta\,\Phi_{\text{nor}}^{-1}(p) + \sqrt{4 + \left[\beta\,\Phi_{\text{nor}}^{-1}(p)\right]^2}\right\}^2.$$

Birnbaum–Saunders Distribution–Continued

To isolate the scale parameter θ and the unitless shape parameter β, we write the cdf and pdf as follows

$$
\begin{aligned}
F_T(t;\beta,\theta) &= \Phi_{\text{lbisa}}\left[\log(t/\theta);\beta\right] \\
f_T(t;\beta,\theta) &= \frac{1}{t}\,\phi_{\text{lbisa}}\left[\log(t/\theta);\beta\right]
\end{aligned}
$$

where

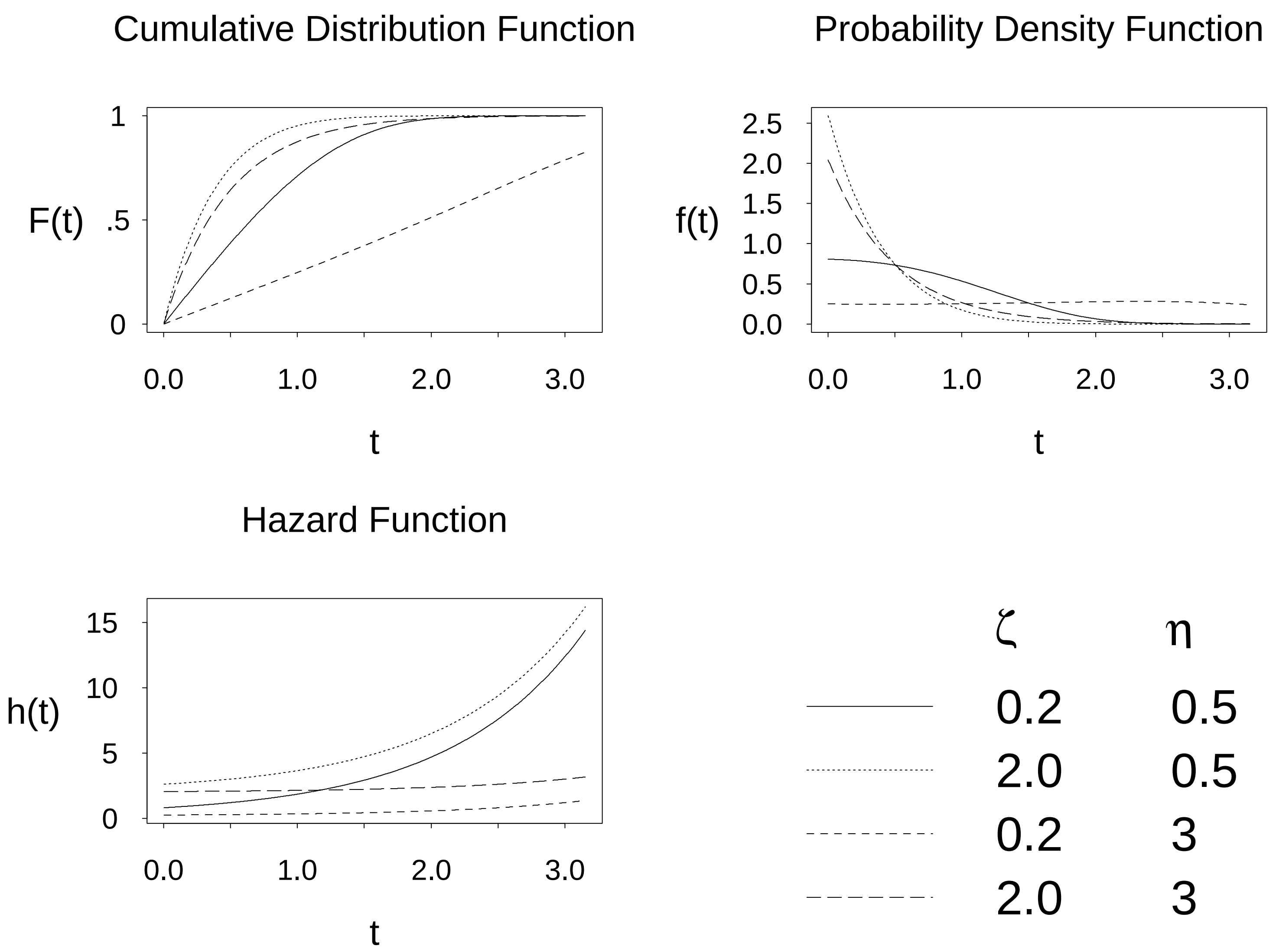
$$
\begin{aligned}
\Phi_{\text{lbisa}}(z;\beta) &= \Phi_{\text{nor}}(\nu) \\
\phi_{\text{lbisa}}(z;\beta) &= \left[\frac{\exp(z/2)+\exp(-z/2)}{2\beta}\right]\phi_{\text{nor}}(\nu)\,,\ -\infty < z < \infty \\
\nu &= \frac{1}{\beta}\left[\exp(z/2)-\exp(-z/2)\right].
\end{aligned}
$$

Birnbaum–Saunders Distribution–Continued

Notes:

- If $T \sim \text{BISA}(\theta, \beta)$ and $c > 0$ then $cT \sim \text{BISA}(c\theta, \beta)$.
- If $T \sim \text{BISA}(\theta, \beta)$ then $1/T \sim \text{BISA}(\theta^{-1}, \beta)$.
- The hazard function BISA $h(t; \theta, \beta)$ is not always increasing.
  - $h(0; \theta, \beta) = 0$.
  - $\lim_{t \to \infty} h(t; \theta, \beta) = 1/(2\theta\beta^2)$.
  - extensive numerical experiments indicate that $h(t; \theta, \beta)$ is always unimodal.
- This distribution was derived by Birnbaum and Saunders (1969) in the modeling of fatigue crack extension.

Examples of Gompertz-Makeham Distributions



Gompertz–Makeham Distribution

- A common parameterization for this distribution is
$$\Pr(T \leq t; \gamma, \kappa, \lambda) \;=\; 1 - \exp\left[-\frac{\lambda\kappa t + \gamma\exp(\kappa t) - \gamma}{\kappa}\right], \quad t > 0.$$
$\gamma > 0, \kappa > 0, \lambda \geq 0$ and all the parameters have units that are the reciprocal of the units of t.

- This distribution originated from the need of a positive random variable with a hazard function similar to the hazard of the SEV. It can be shown that
$$\Pr(T \leq t; \gamma, \kappa, \lambda) \;=\; 1 - \left[\frac{1 - \Phi_{\text{sev}}\left(\frac{t-\mu}{\sigma}\right)}{1 - \Phi_{\text{sev}}\left(\frac{-\mu}{\sigma}\right)}\right]\exp(-\lambda t)$$
where $\mu = -(1/\kappa)\log(\gamma/\kappa)$, $\sigma = 1/\kappa$.

- When $\lambda = 0$, one gets Gompertz–distribution which corresponds to a truncated SEV at the origin.

Gompertz–Makeham Continued

The parameterization in terms of $[\theta, \psi, \eta] = [1/\kappa, \log(\kappa/\gamma), \lambda/\kappa]$ isolates the scale parameter from the shape parameter and we say that $T \sim \text{GOMA}(\theta, \psi, \eta)$, if

$$
\begin{aligned}
F_T(t; \theta, \psi, \eta) &= \Phi_{\text{lgoma}}[\log(t/\theta); \psi, \eta] \\
f_T(t; \theta, \psi, \eta) &= \frac{1}{t}\,\phi_{\text{lgoma}}[\log(t/\theta); \psi, \eta] \\
h_T(t; \theta, \psi, \eta) &= \frac{\eta}{\theta} + \frac{\exp(-\psi)}{\theta}\exp\left(\frac{t}{\theta}\right), \quad t > 0
\end{aligned}
$$

here θ is a scale parameter, ψ and η are unitless shape parameters, and

$$
\begin{aligned}
\Phi_{\text{lgoma}}(z; \psi, \eta) &= 1 - \exp\left\{\exp(-\psi) - \exp\left[\exp(z) - \psi\right] - \eta\,\exp(z)\right\} \\
\phi_{\text{lgoma}}(z; \psi, \eta) &= \exp(z)\left\{\eta + \exp\left[\exp(z) - \psi\right]\right\}\left[1 - \Phi_{\text{lgoma}}(z; \psi, \eta)\right]
\end{aligned}
$$

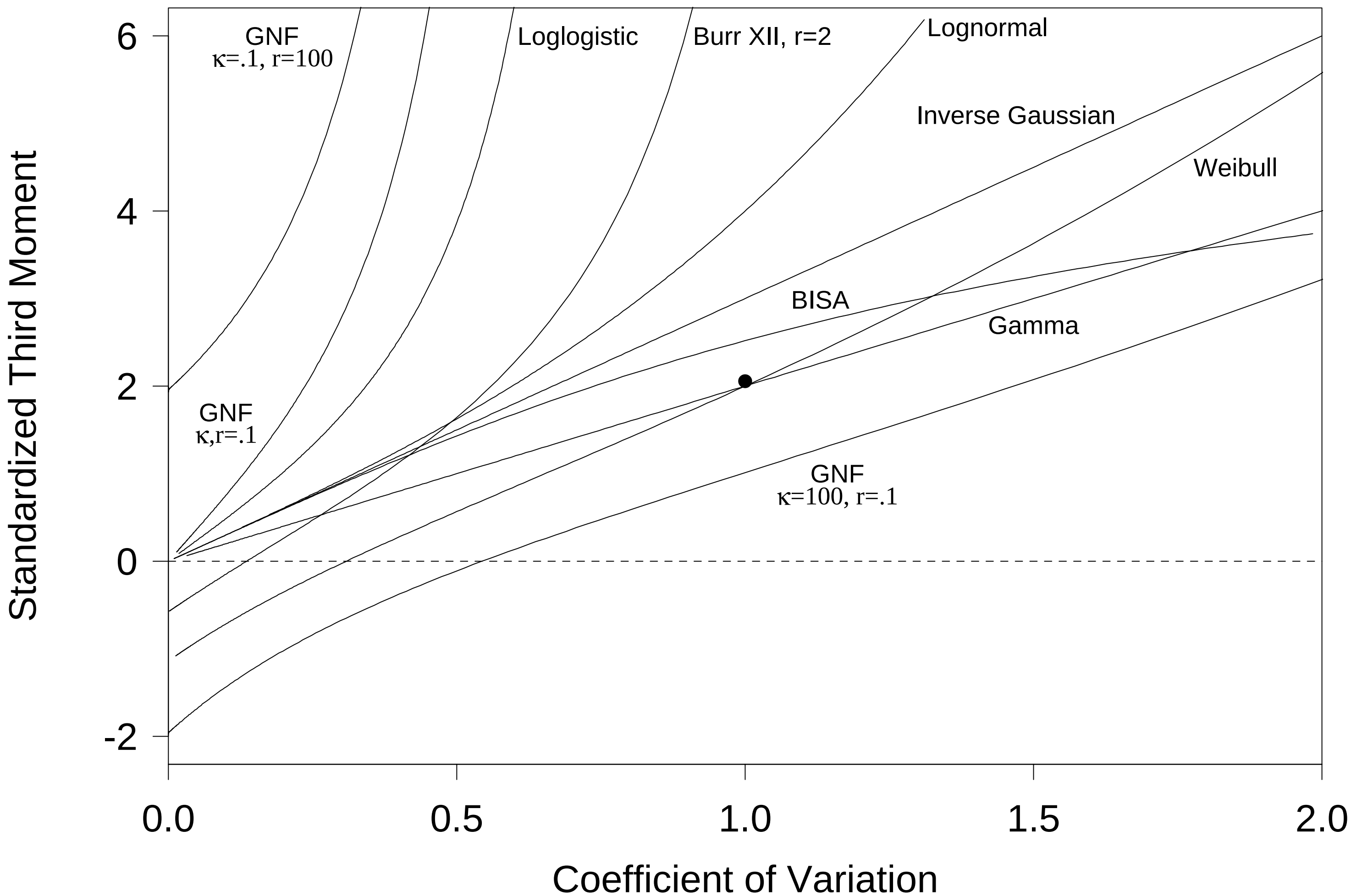
are, respectively, the standardized cdf and pdf of $Z = \log(t/\theta)$.

Gompertz–Makeham Distribution–Continued

Notes:

- $h_T(0; \theta, \psi, \eta) = (1/\theta)[\eta + \exp(-\psi)]$.
- $h_T(t; \theta, \psi, \eta)$ increases with t at an exponential rate.
- If $T \sim \text{GOMA}(\theta, \psi, \eta)$ and $c > 0$ then $cT \sim \text{GOMA}(c\theta, \psi, \eta)$.

Standardized Third Moment Versus Coefficient of Variation



Comparison of Spread and Skewness Parameters

- The **standardized** third central moment of T defined by

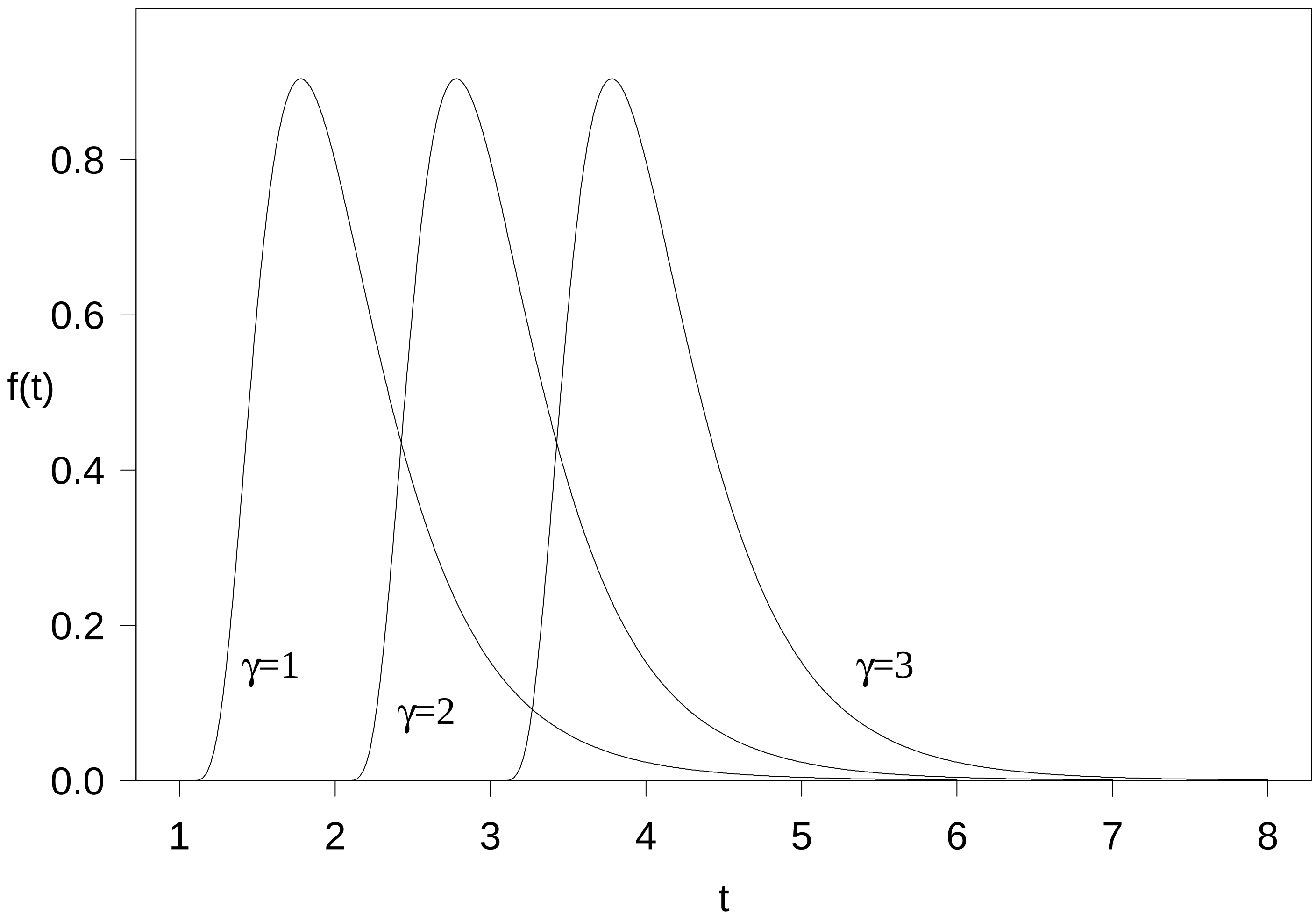
$$\gamma_3 = \frac{\int_0^\infty [t - \mathsf{E}(T)]^3 f(t;\boldsymbol{\theta})\,dt}{[\mathsf{Var}(T)]^{\frac{3}{2}}}$$

  is a measure of the skewness in the distribution of T. This parameter is unitless and it has the these properties:

  - ▶ Distributions with $\gamma_3 > 0$ will tend to be skewed to the right.
  - ▶ Distributions with $\gamma_3 < 0$ will tend to be skewed to the left (e.g., the Weibull distribution with large β).

- The unitless **coefficient** of variation of T, $\gamma_2 = \sqrt{\mathsf{Var}(T)}/\mathsf{E}(T)$, is useful for comparing the relative amount of variability in the distributions of random variables having different units.

pdfs for Three-Parameter Lognormal Distributions for $\mu = 0$ and $\sigma = .5$ with $\gamma =$ 1,2,3.



Distributions with a Threshold Parameter

- So far we have discussed nonnegative random variables with cdfs that begin increasing at $t = 0$.
- One can generalize these and similar distributions by adding a **threshold**, γ, to shift the beginning of the distribution away from 0.
- Distributions with a threshold are particularly useful for fitting skewed distributions that are shifted far to the right of 0.
- The cdf for location-scale log-based threshold distributions is

$$\begin{aligned} F(t;\mu,\sigma,\gamma) &= \Phi\left[\frac{\log(t-\gamma)-\mu}{\sigma}\right] \\ \text{or} \quad F(t;\eta,\sigma,\gamma) &= \Phi\left[\log\left(\frac{t-\gamma}{\eta}\right)^{1/\sigma}\right], \quad t > \gamma \end{aligned}$$

where $\eta = \exp(\mu)$, $-\infty < \gamma < \infty$, $-\infty < \mu < \infty$, $\sigma > 0$, $\eta > 0$, and Φ is a completely specified cdf.

Examples of Distributions with a Threshold Parameter

- Three-parameter lognormal distribution

$$F(t;\mu,\sigma,\gamma) = \Phi_{\text{nor}}\left[\frac{\log(t-\gamma)-\mu}{\sigma}\right],\ t>\gamma.$$

- Three-parameter Weibull distribution

$$\begin{aligned} F(t;\eta,\beta,\gamma) &= 1-\exp\left[-\left(\frac{t-\gamma}{\eta}\right)^{\beta}\right] \\ &= \Phi_{\text{sev}}\left[\frac{\log(t-\gamma)-\mu}{\sigma}\right],\ t>\gamma \end{aligned}$$

where $\sigma = 1/\beta$ and $\mu = \log(\eta)$.

Properties of Distributions with a Threshold

- When the distribution of T has a threshold, γ, then the distribution of $W = T - \gamma$ has a distribution with 0 threshold.
- The properties of the distribution of T are **closely** related to the properties of the distribution of W.
- In general, $\mathsf{E}(T) = \gamma + \mathsf{E}(W)$ and $t_p = \gamma + w_p$, where w_p is the p quantile of the distribution of W.
- Changing γ simply shifts the distribution on the time axis, there is no effect on the distribution's spread or shape. Thus $\mathsf{Var}(T) = \mathsf{Var}(W)$.
- There are, however, some very specific issues in the estimation of γ because the points at which the cdf is positive depends on γ.

Embedded Models

- For some values of (μ, σ, γ), the model is very similar to a two-parameter location-scale model, as described below.

- **Embedded models:** Using the **reparameterization**, $\alpha = \gamma + \eta$, $\varsigma = \sigma\eta$, the model becomes

$$\begin{aligned} F(t; \alpha, \sigma, \varsigma) &= \Phi\left[\log\left(1 + \sigma \times \frac{t - \alpha}{\varsigma}\right)^{1/\sigma}\right] \\ &= \Phi\left[\log\left(1 + \sigma z\right)^{1/\sigma}\right], \quad \text{for } z > -1/\sigma \end{aligned}$$

  where $z = (t - \alpha)/\varsigma$.

  When $\sigma \to 0^{+}$, $(1 + \sigma z)^{1/\sigma} \to \exp(z)$, and the **limiting** distribution is

$$F(t; \alpha, 0, \varsigma) = \Phi\left(z\right), \quad \text{for } -\infty < t < \infty.$$

- For example, if $\Phi = \Phi_{\text{sev}}$ the limiting distribution is the SEV and if $\Phi = \Phi_{\text{nor}}$ the limiting distribution is normal.

Some Comments on the Embedded Models

- The limiting distribution arises when

  a. $1/\sigma$ and η are going to ∞ at the same rate, and

  b. γ is going to $-\infty$ at the same rate that η is going to ∞.

- Precisely, if $F(t; \eta_i, \sigma_i, \gamma_i)$ is a sequence of cdfs such that

$$\begin{aligned} \sigma_i &\to 0 \\ \varsigma &= \lim_{i\to\infty} (\sigma_i \eta_i) \quad \text{with } 0 < \varsigma < \infty \\ \alpha &= \lim_{i\to\infty} (\gamma_i + \eta_i) \quad \text{with } -\infty < \alpha < \infty \end{aligned}$$

  then $F(t; \eta_i, \sigma_i, \gamma_i) \to \Phi(z)$, where $z = (t - \alpha)/\varsigma$

Generalized Threshold Scale (GETS) Models

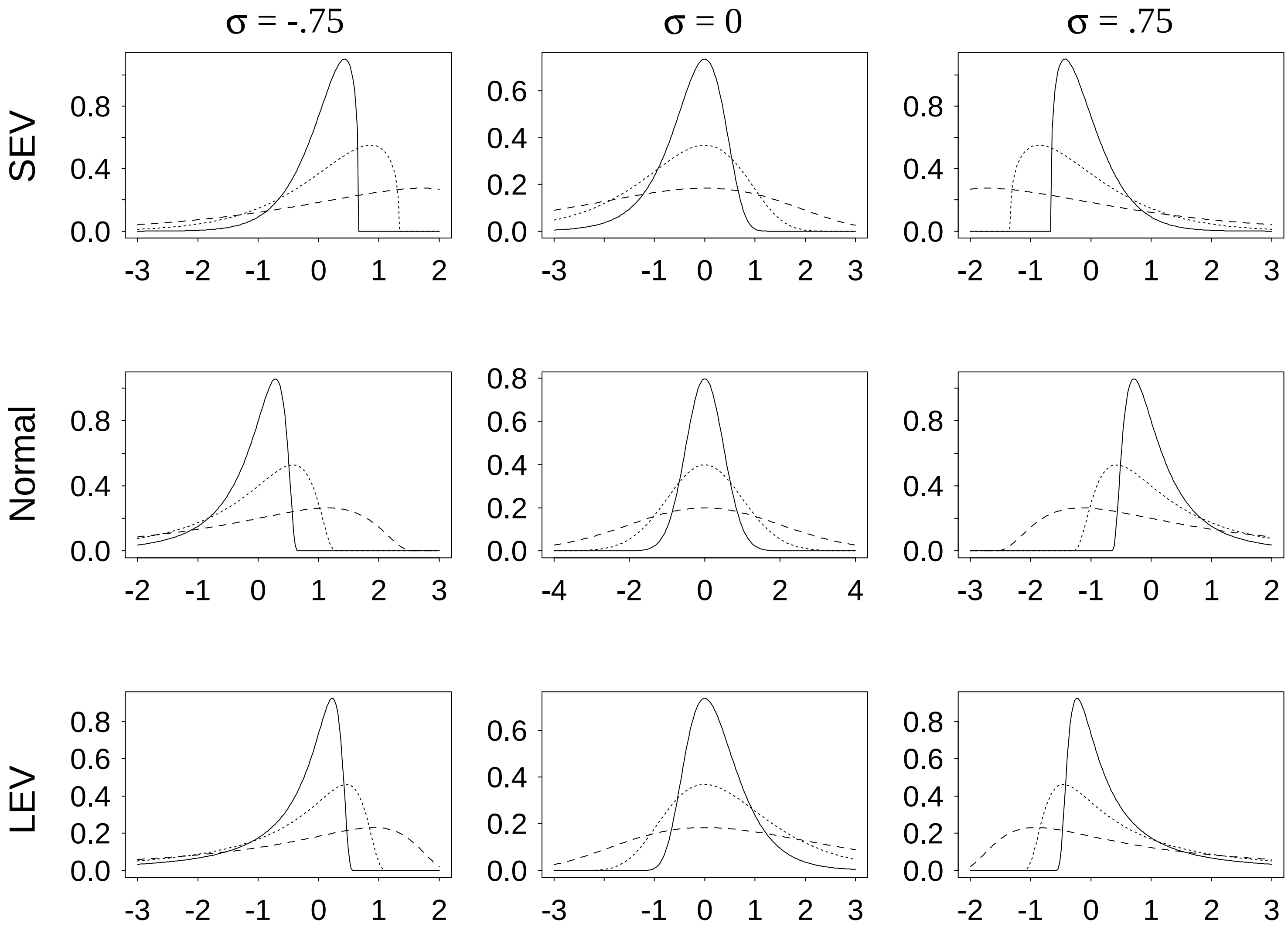
- The original threshold parameter space $(\alpha, \sigma, \varsigma)$ (with $\sigma > 0$) does not contain the limiting distributions.

- It is convenient to enlarge the parameter space such that the limiting distributions are interior points of the parameter space.

- This is achieved by allowing σ to take values in $(-\infty, \infty)$.

- The family of distributions corresponding to this enlarged parameter space is known as the generalized threshold scale (GETS) family .

SEV-GETS, NOR-GETS, and LEV-GETS pdfs with $\alpha = 0$, $\sigma = -.75, 0, .75$, and $\varsigma = .5$ (Least Disperse), 1, and 2 (Most Disperse)



GETS **MODEL**

- The **cdf** for the GETS model is

$$F(t;\alpha,\sigma,\varsigma) = \begin{cases} \Phi\left[\log\left(1+\sigma z\right)^{1/\sigma}\right], & \text{for } \sigma > 0,\ z > -1/\sigma \\ \Phi\left(z\right), & \text{for } \sigma = 0,\ -\infty < t < \infty \\ 1 - \Phi\left[\log\left(1+\sigma z\right)^{1/|\sigma|}\right], & \text{for } \sigma < 0,\ z < -1/\sigma \end{cases}$$

where $z = (t-\alpha)/\varsigma$.

- The corresponding **pdf** is

$$f(t;\alpha,\sigma,\varsigma) = \begin{cases} \phi\left[\log\left(1+\sigma z\right)^{1/|\sigma|}\right] \times \frac{1}{\varsigma(1+\sigma z)}, & \text{for } \sigma \neq 0 \\ \phi\left(z\right) \times \frac{1}{\varsigma}, & \text{for } \sigma = 0,\ -\infty < t < \infty \end{cases}$$

Note: for $\sigma > 0, z > -1/\sigma$ and for $\sigma < 0, z < -1/\sigma$.

- If $T \sim$ GETS$(\alpha, \sigma, \varsigma)$ and $a \neq 0$ then $(aT + b) \sim$ GETS$(a\alpha + b, a\sigma/|a|, \varsigma|a|)$.

Some Special Cases

- The GETS model includes all the location-scales distributions. These are obtained when $\sigma = 0$, as
$$F(t;\alpha,0,\varsigma) = \Phi[(t-\alpha)/\varsigma].$$
This includes the normal, logistic, SEV, LEV, etc.

- The GETS includes all the threshold, log-based location-scale distributions. These are obtained with $\sigma > 0$ which gives
$$F(t;\alpha,\sigma,\varsigma) = \Phi\{[\log(t-\gamma)-\mu]/\sigma\}, \quad t > \gamma$$
where $\gamma = \alpha - \varsigma/\sigma$, $\mu = \log(\varsigma/\sigma)$.

  - With $\Phi = \Phi_{\text{nor}}$ this gives the lognormal with a threshold.

  - With $\Phi = \Phi_{\text{sev}}$ this gives the Weibull (also known as Weibull-type for **minima**) with a threshold.

  - And with $\Phi = \Phi_{\text{lev}}$ one obtains the Fréchet for **maxima** with a threshold.

Some Special Cases-Continued

- The GETS includes the reflection (negative) of the threshold, log-based location-scale distributions. These are obtained with $\sigma < 0$, giving

$$F(t;\alpha,\sigma,\varsigma) = \Phi\{[\log(-t-\gamma)-\mu]/\sigma\}, \quad t < -\gamma$$

  where $\gamma = -(\alpha - \varsigma/\sigma)$, $\mu = \log(-\varsigma/\sigma)$.

- With $\Phi = \Phi_{\text{nor}}$ this gives the negative of a lognormal with a threshold.

- With $\Phi = \Phi_{\text{sev}}$ this gives the negative of a Weibull with a threshold. Or equivalently a Weibull-type distribution for **maxima**.

- With with $\Phi = \Phi_{\text{lev}}$ one obtains the negative of a Fréchet for **maxima** with a threshold. Or equivalently, a Fréchet-type distribution for **minima**.

Quantiles for the GETS Distribution

- **Quantiles:** the p quantile of the GETS distribution is

$$t_p \; = \; \alpha + \varsigma \times w(\sigma, p)$$

where

$$w(\sigma, p) \; = \; \begin{cases} \frac{\exp[\sigma\Phi^{-1}(p)]-1}{\sigma}, & \text{for } \sigma > 0 \\ \Phi^{-1}(p), & \text{for } \sigma = 0 \\ \frac{\exp\{|\sigma|\Phi^{-1}(1-p)\}-1}{\sigma}, & \text{for } \sigma < 0 \end{cases}$$

- Then for fixed σ, t_p versus $w(\sigma, p)$ plots as a straight line.

GETS **Stable Parameterization**

- **Parameterization for Numerical Stability:** with $p_1 < p_2$, a stable parameterization can be obtained using two quantiles and σ, i.e., $(t_{p_1}, t_{p_2}, \sigma)$.

- Using the expression for the quantiles

$$\begin{aligned} t_{p_1} &= \alpha + \varsigma \times w(\sigma, p_1) \\ t_{p_2} &= \alpha + \varsigma \times w(\sigma, p_2). \end{aligned}$$

Solving for α and ς

$$\begin{aligned} \alpha &= \frac{w(\sigma, p_1) \times t_{p_2} - w(\sigma, p_2) \times t_{p_1}}{w(\sigma, p_1) - w(\sigma, p_2)} \\ \varsigma &= \frac{t_{p_1} - t_{p_2}}{w(\sigma, p_1) - w(\sigma, p_2)}. \end{aligned}$$

Finite (Discrete) Mixture Distributions

- The cdf of units in a population consisting of a mixture of units from k different populations can be expressed as

$$F(t;\boldsymbol{\theta}) = \sum_i \xi_i F_i(t;\boldsymbol{\theta}_i)$$

  where $\boldsymbol{\theta} = (\boldsymbol{\theta}_1, \boldsymbol{\theta}_2, \ldots, \xi_1, \xi_2, \ldots)$, $\xi_i \geq 0$, and $\sum_i \xi_i = 1$.

- Mixtures tend to have a large number of parameters and estimation can be complicated. But estimation is facilitated by:

  - identification of the individual population from which sample units originated.

  - considerable **separation** in the components and/or enormous amounts of data.

- Sometimes it is sufficient to fit a simpler distribution to describe the overall mixture.

Continuous Mixture (Compound Distributions)

- These probability models arise from distributions in which one or more of the parameters are continuous random variable.

- These distributions are called **compound** distributions and correspond to continuous mixture of a family of distributions, as follows:

  Assume that for a fixed value of a scalar parameter θ_1, $T|\theta_1 \sim f_{T|\theta_1}(t;\boldsymbol{\theta})$ with $\boldsymbol{\theta} = (\theta_1, \boldsymbol{\theta}_2)$. Assuming that θ_1 is random from unit to unit with $\theta_1 \sim f_{\theta_1}(\vartheta;\boldsymbol{\theta}_3)$, where $\boldsymbol{\theta}_3$ does not have elements in common with $\boldsymbol{\theta}$, then

$$\begin{aligned} F(t;\boldsymbol{\theta}_2,\boldsymbol{\theta}_3) = \Pr(T \le t) &= \int_{-\infty}^{\infty} \Pr(T \le t|\theta_1 = \vartheta) f_{\theta_1}(\vartheta;\boldsymbol{\theta}_3) d\vartheta \\ &= \int_{-\infty}^{\infty} F_{T|\theta_1=\vartheta}(t;\boldsymbol{\theta}) f_{\theta_1}(\vartheta;\boldsymbol{\theta}_3) d\vartheta \end{aligned}$$

  and the corresponding pdf is

$$f(t;\boldsymbol{\theta}_2,\boldsymbol{\theta}_3) = \int_{-\infty}^{\infty} f_{T|\theta_1=\vartheta}(t;\boldsymbol{\theta}) f_{\theta_1}(\vartheta;\boldsymbol{\theta}_3) d\vartheta.$$

Pareto Distribution as a Compound Distribution

- If life of the the ith unit in a population can be modeled by

$$T|\eta \sim \mathrm{EXP}(\eta).$$

- But the failure rate varies from unit to unit in the population according to a GAM(θ, κ), i.e,

$$\frac{1}{\eta} \sim \mathrm{GAM}(\theta, \kappa).$$

- Then the unconditional failure time of a unit selected at random from the population follows a Pareto distribution of the form

$$F(t;\theta,\kappa) = 1 - \frac{1}{(1+\theta t)^{\kappa}}, \quad t > 0.$$

Other Distributions

- Power distributions.
- Distributions based on stochastic components of physical/chemical degradation models.
- Multivariate failure time distributions.