Chapter 13

Degradation Data, Models, and Data Analysis

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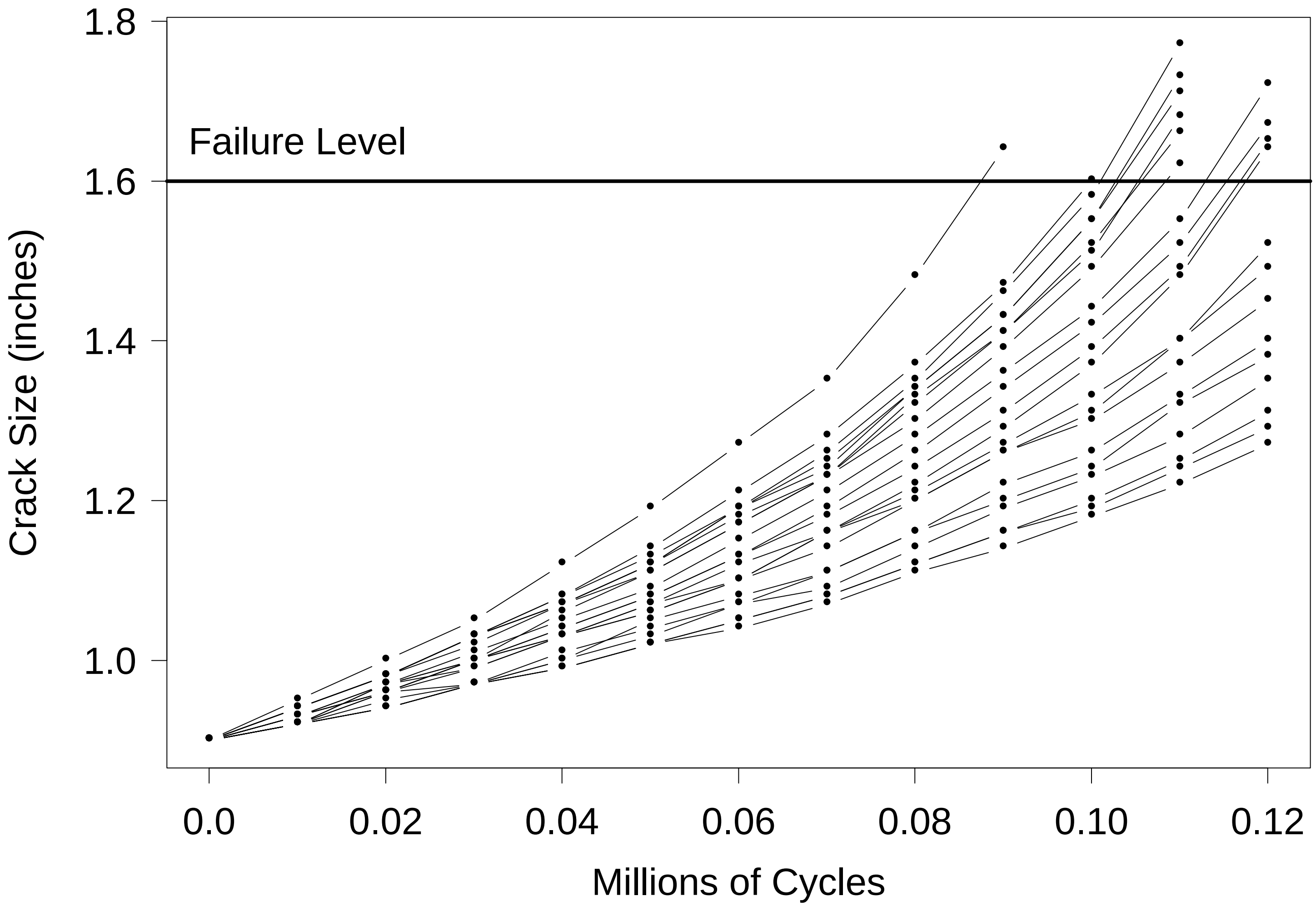
Degradation Data, Models, and Data Analysis
Chapter 13 Objectives

- Describe a number of useful degradation reliability models.
- Show the connection between degradation reliability models and failure-time reliability models.
- Show how degradation measures, when available, can be used to advantage in estimating reliability.
- Present methods of data analysis and reliability inference for degradation data.
- Compare degradation data analysis with traditional failure time data analysis.

Fatigue Crack Size Observations for Alloy-A (Bogdanoff & Kozin 1985)



Alloy-A Fatigue Crack-Size Data

- Data from Hudak, Saxena, Bucci, and Malcolm (1978) and Bogdanoff and Kozin (1985, page 242).

- Suppose investigators wanted to:
  - Estimate materials-related crack growth parameters.
  - Estimate time (measured in number of cycles) at which 50% of the cracks would reach 1.6 inches.
  - Assess adequacy of the **paris** model.

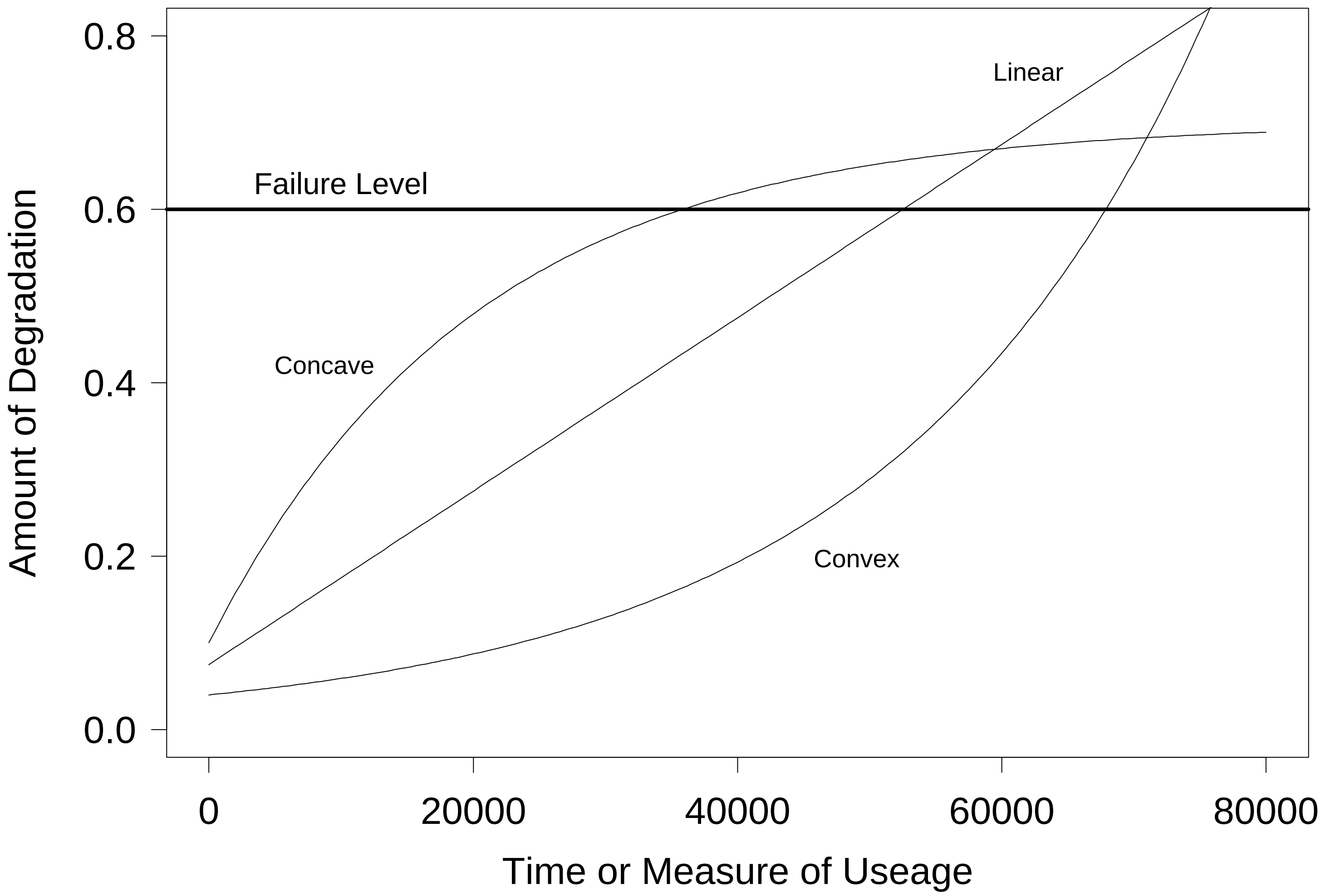
Degradation Data

- Sometimes possible to measure degradation directly over time
  - ▶ Continuously.
  - ▶ At specific points in time.
- Degradation is natural response for some tests.
- Degradation data can provide considerably more reliability information than censored failure-time data (especially with few or no failures).
- Direct observation of the degradation process allows direct modeling of the failure-causing mechanism.

Degradation Leading to Failure

- Most failures can be traced to an underlying degradation process.
- Degradation curves can have different shapes.
- Failure occurs when degradation crosses a threshold .
- Some applications have more than one degradation variable or more than one underlying degradation process.
- Examples here have only one degradation variable and underlying degradation process.

Possible Shapes for Univariate Degradation Curves



Possible Shapes for Univariate Degradation Curves

- **Linear degradation:** Degradation **rate**

$$\frac{d\,\mathcal{D}(t)}{d\,t} = C$$

is constant over time. Degradation **level** at time t, $\mathcal{D}(t) = \mathcal{D}(0) + C \times t$, is linear in t. Examples include: amount of automobile tire tread wear and mechanical wear on a bearing.

- **Concave degradation:** Degradation rate decreasing in time. Degradation level increasing at a decreasing rate. Examples include chemical processes with a limited amount of material to react.

- **Convex degradation:** Degradation rate increasing in time. Degradation level increasing at an increasing rate. Examples include crack growth.

Paris Crack Growth Model

- The paris model is

$$\frac{d\,a(t)}{d\,t} = C \times [\Delta K(a)]^m$$

  is a commonly used model to describe the growth of fatigue cracks over some range of size.

- $C > 0$ and $m > 0$ are materials properties

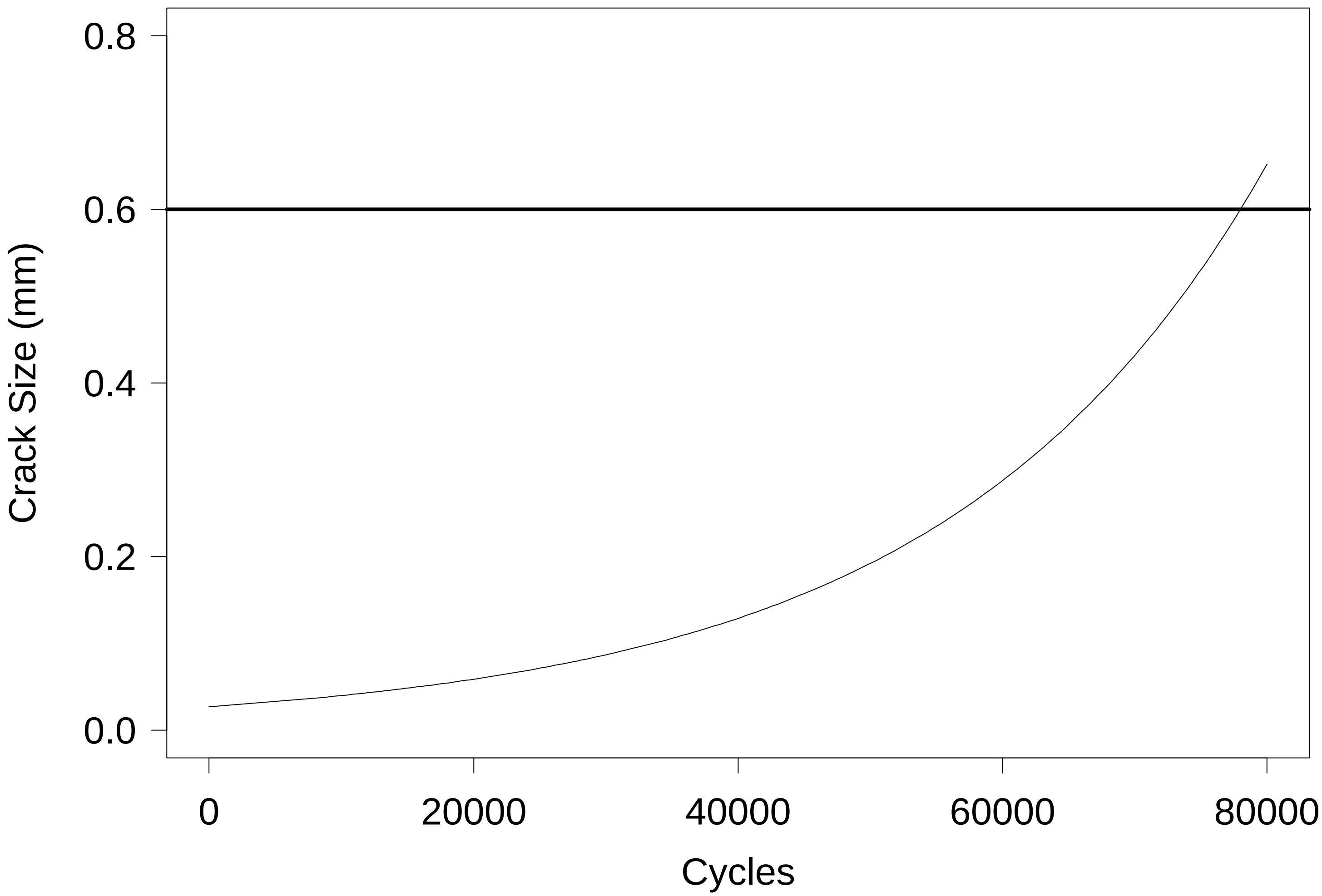
- $\Delta K(a)$, the stress intensity function of a. Form of $K(a)$ depends on applied stress, part dimensions, and geometry.

- To model a two-dimensional edge-crack in a plate with a crack that is small relative to the width of the plate (say less than 3%), $K(a) = \text{Stress}\sqrt{\pi a}$ and the solution to the resulting differential equation is

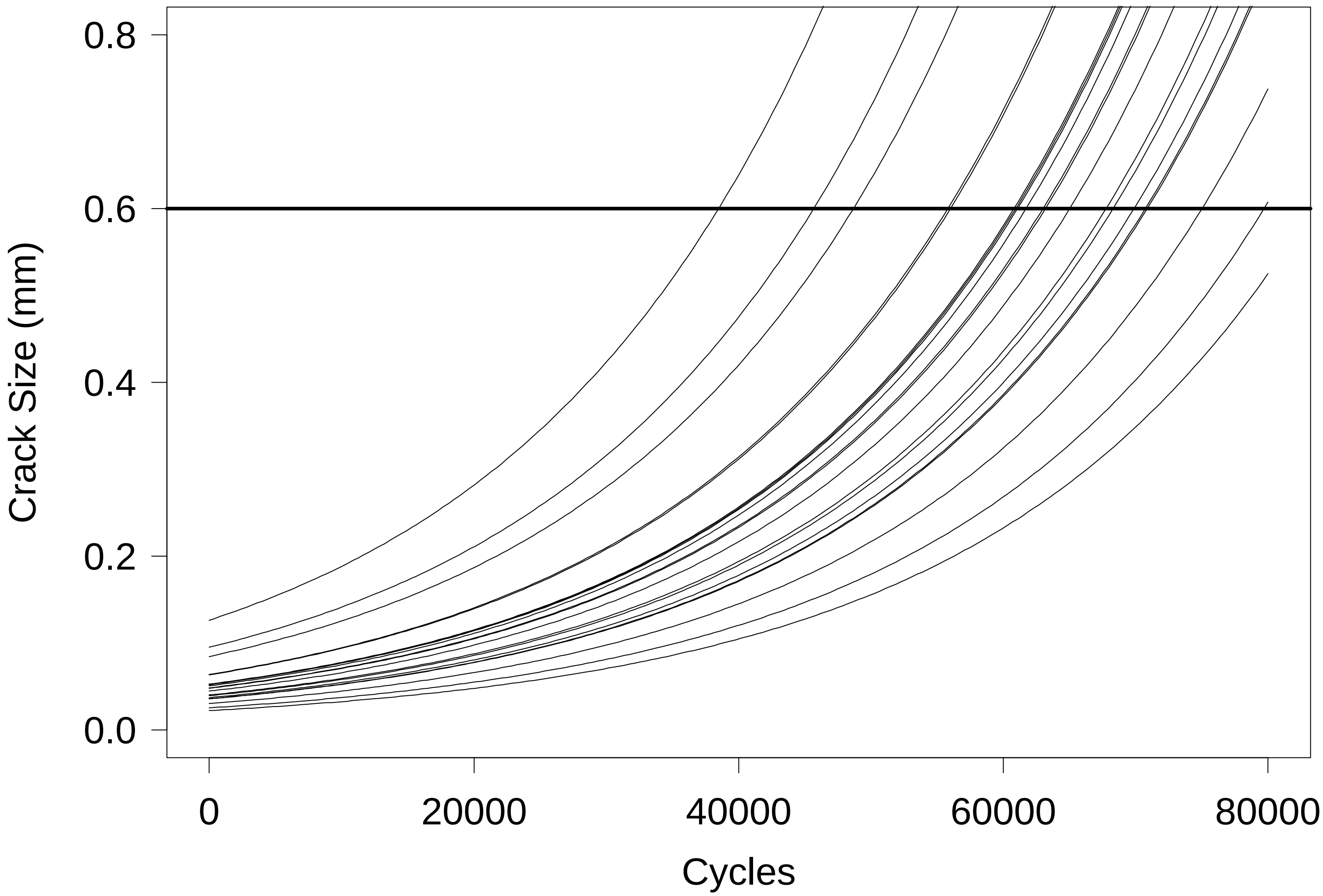
$$a(t) = \begin{cases} \left[\{a(0)\}^{1-m/2} + (1-m/2) \times C \times (\text{Stress}\sqrt{\pi})^m \times t\right]^{\dfrac{2}{2-m}}, & m \neq 2 \\ a(0) \times \exp\left[C \times (\text{Stress}\sqrt{\pi})^2 \times t\right], & m = 2 \end{cases}$$

Paris Model with no Variability

$$\frac{d\,a(t)}{d\,t} = C \times [\Delta K(a)]^{m}$$



Paris Model with Unit-to-Unit Variability in Initial Crack Size but with Fixed Materials Parameters and Constant Stress

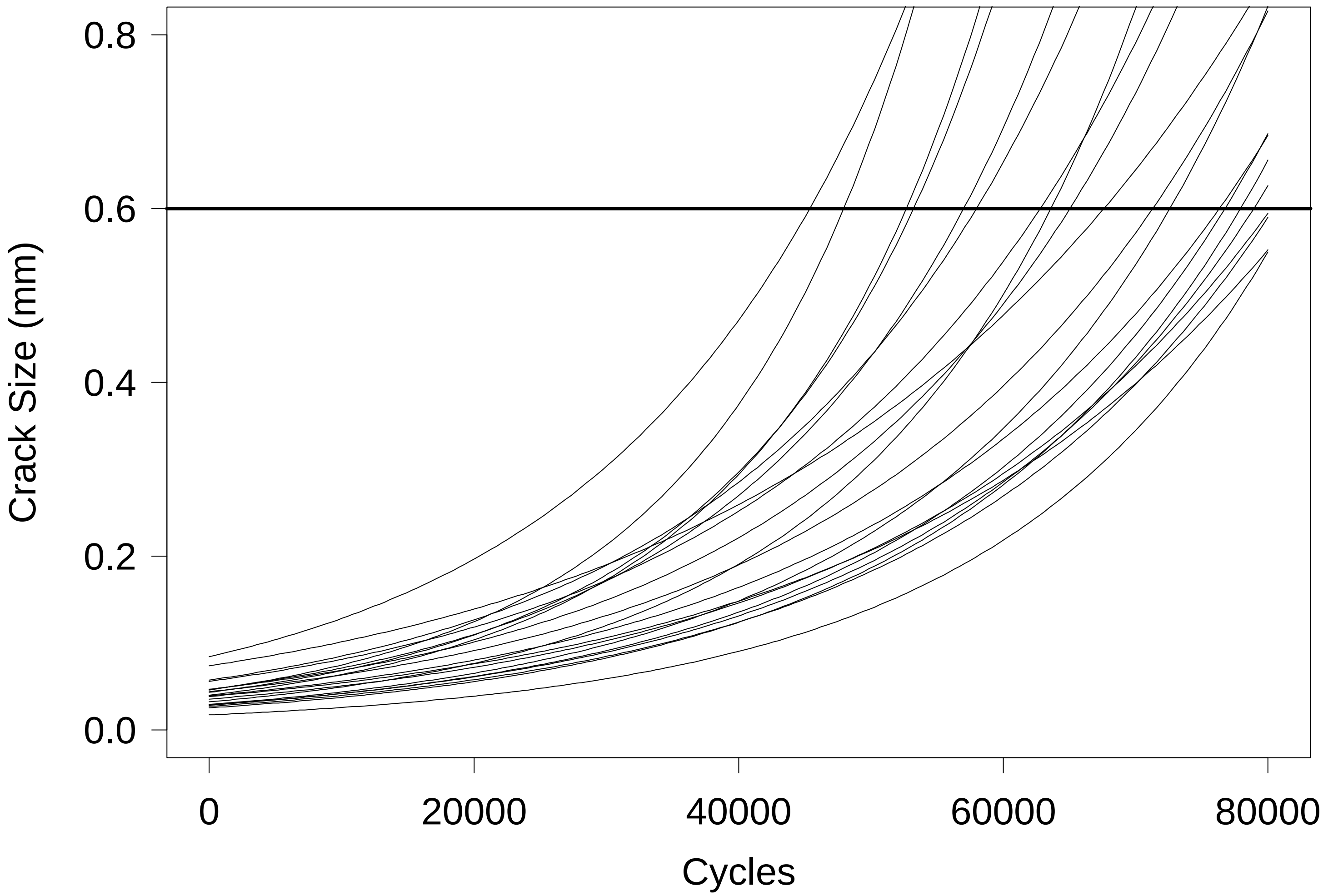


Models for Variation in Degradation and Failure Time

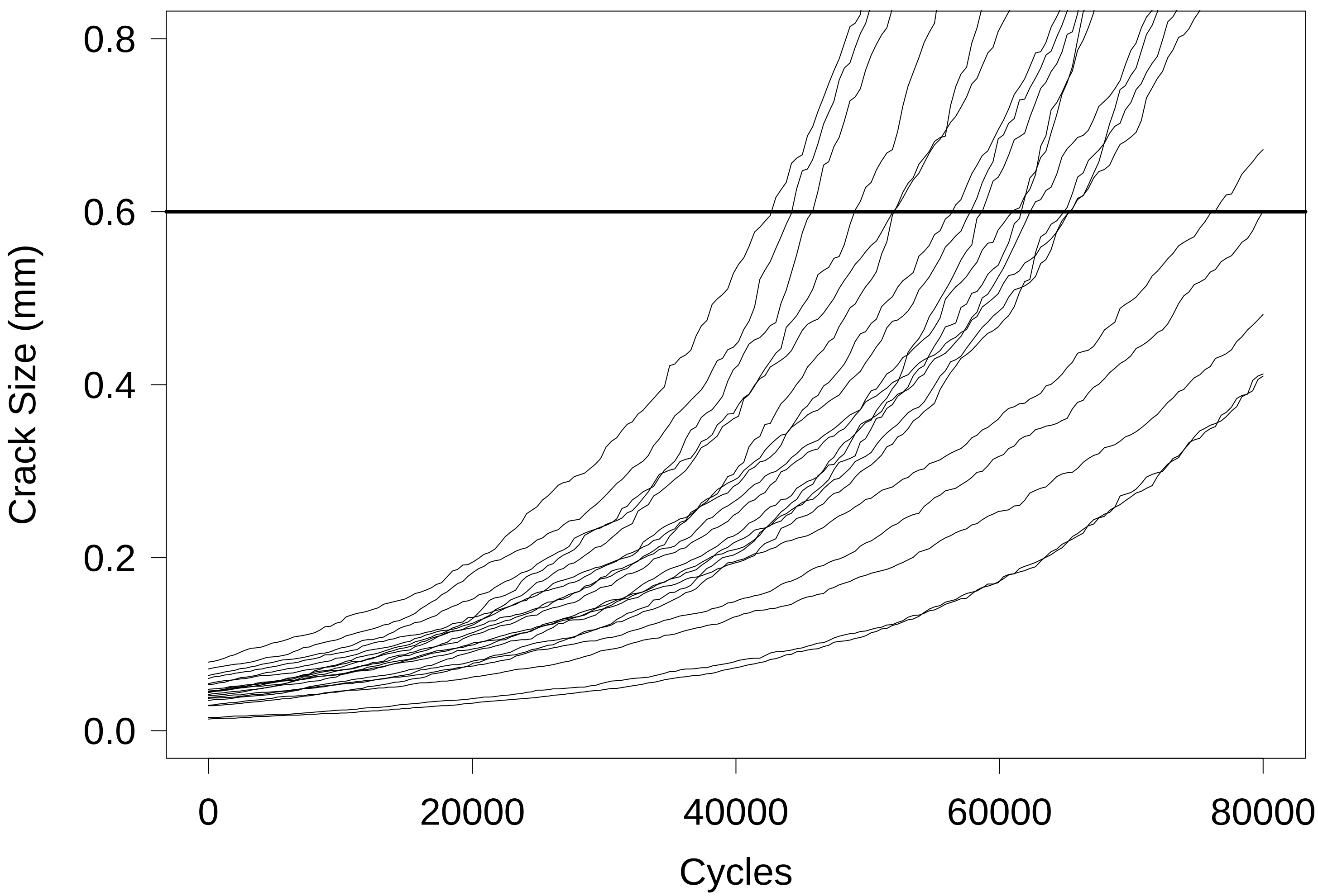
If all manufactured units were identical, operated at exactly the same time, under exactly the same conditions, and in exactly the same environment, and if every unit failed as it reached a particular **critical** level of degradation, then all units would fail at exactly the same time.

- Need to identify and model important sources of variability in the degradation process.
- Quantities that might be modeled as random include:
  - ▶ Initial conditions (flaw size, amount of material).
  - ▶ Materials parameters (related to degradation rate).
  - ▶ Level of degradation at which unit will fail.
- Stochastic process variability (e.g., stress of other environmental variables changing over time).

Paris Model with Unit-to-Unit Variability in the Initial Crack Size and Materials Parameters but Constant Stress



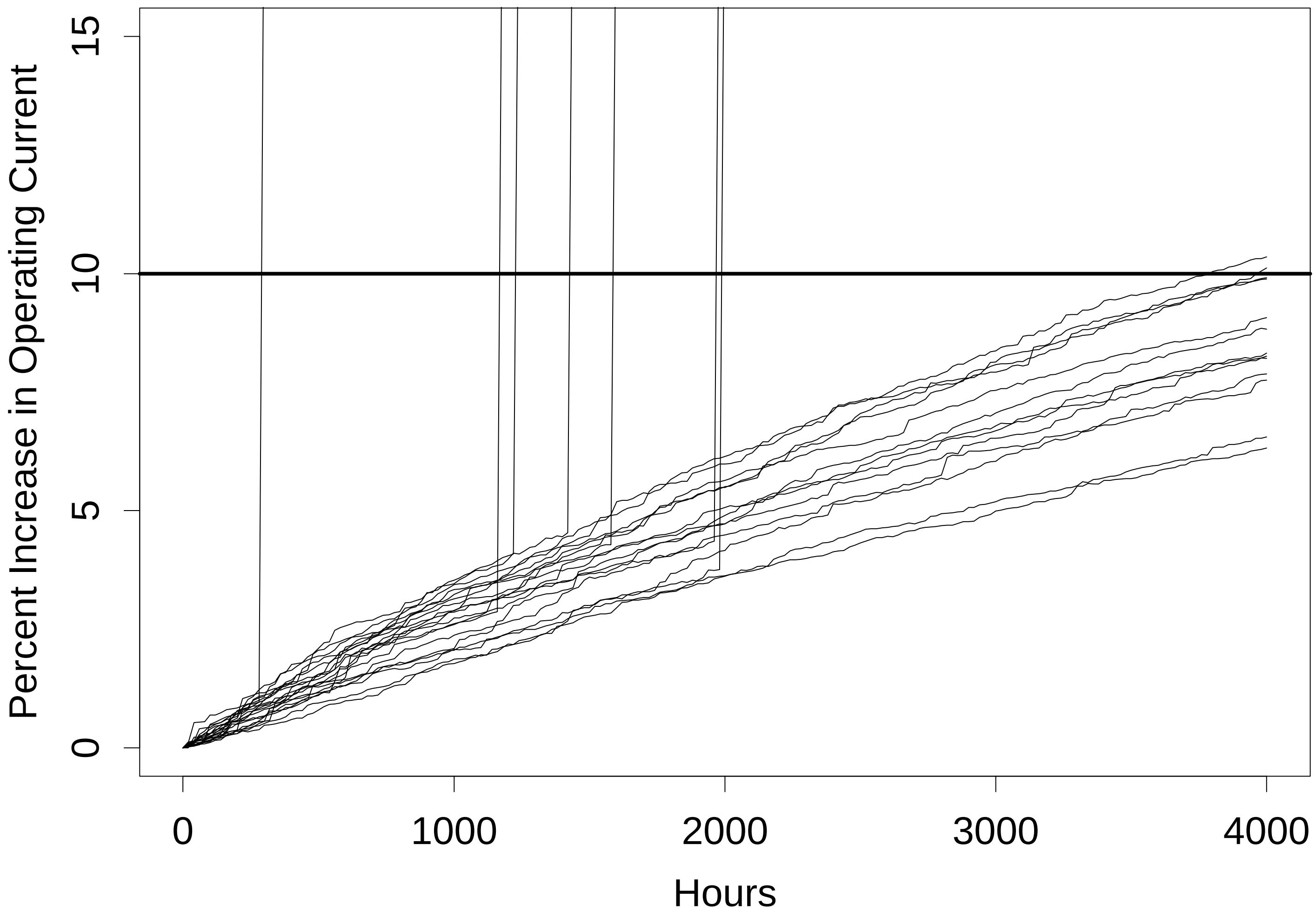
Paris Model with Unit-to-Unit Variability in the Initial Crack Size and Materials Parameters and Stochastic Stress



Limitations of Degradation Data

- Degradation data may be difficult or impossible to obtain (e.g., destructive measurements).
- Obtaining degradation data may have an effect on future product degradation (e.g., taking apart a motor to measure wear).
- Substantial measurement error can diminish the information in degradation data.
- Analyses more complicated; requires statistical methods not yet widely available.
  (Modern computing capabilities should help here)
- Degradation level may not correlate well with failure.

Percent Increase in Operating Current for GaAs Lasers Tested at 80°C



General Degradation Path Model

- $\mathcal{D}_{ij} = \mathcal{D}(t_{ij}, \beta_{1i}, \ldots, \beta_{ki})$ is the degradation path for unit i at time t (measured in hours, cycles, etc.).

- Observed sample degradation path of unit i at time t_j is

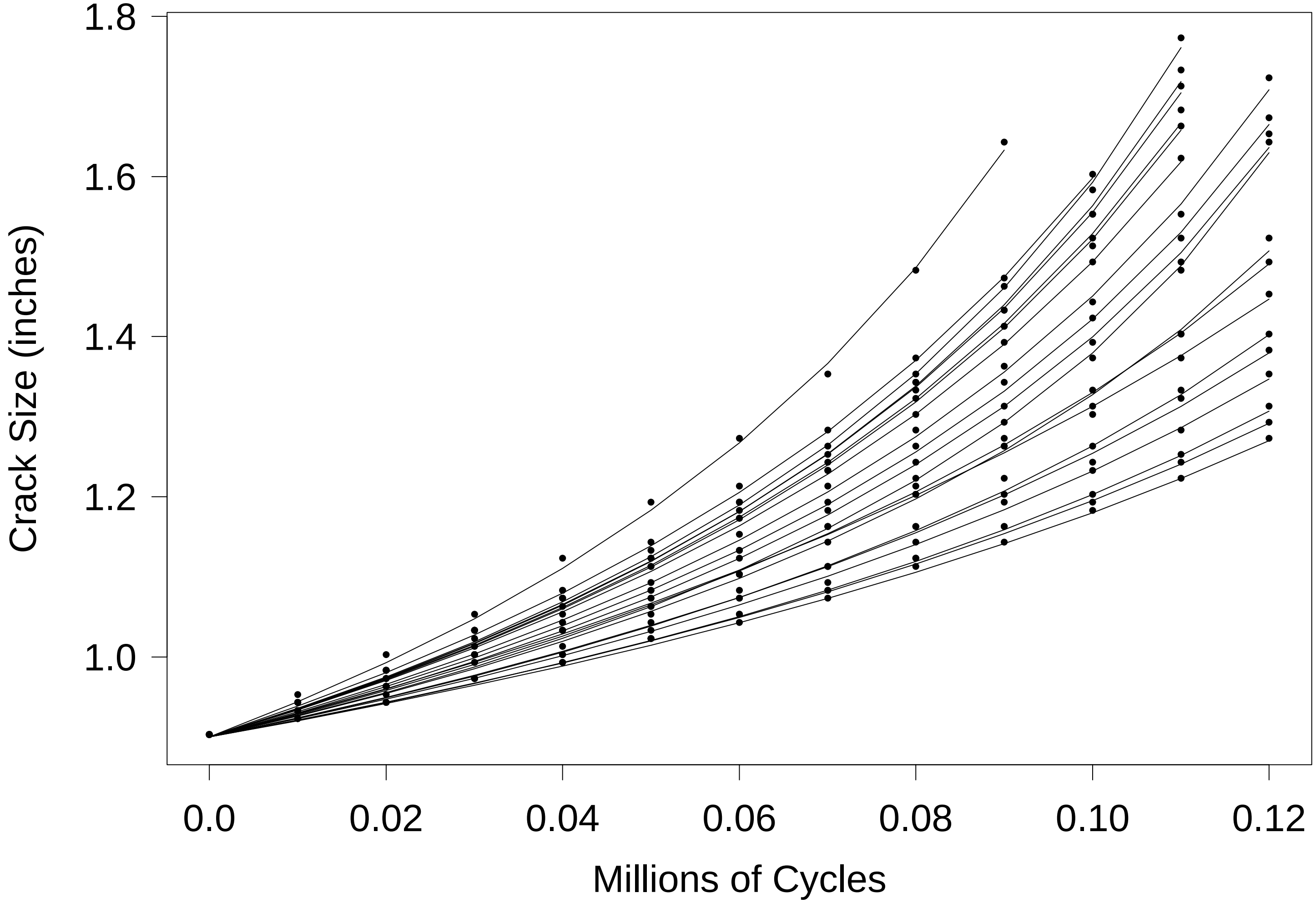
$$y_{ij} = \mathcal{D}_{ij} + \epsilon_{ij}, \quad i = 1, \ldots, n, \quad j = 1, \ldots, m_i$$

- Residuals $\epsilon_{ij} \sim \text{NOR}\,(0, \sigma_\epsilon)$ describe a combination of measurement error and model error.

- For unit i, $\beta_{1i}, \ldots, \beta_{ki}$ is a vector of k unknown parameters.

- Some of the $\beta_{1i}, \ldots, \beta_{ki}$ are random from unit to unit. Model appropriate function of $\beta_{1i}, \ldots, \beta_{ki}$ with multivariate normal distribution (MVN) with parameters $\boldsymbol{\mu}_{\boldsymbol{\beta}}$ and $\Sigma_{\boldsymbol{\beta}}$.

Alloy-A Fatigue Crack Size Observations and Fitted Paris-Rule Model



Estimation of Degradation Model Parameters

- The likelihood for the random-parameter degradation model is $L(\boldsymbol{\mu}_{\boldsymbol{\beta}}, \Sigma_{\boldsymbol{\beta}}, \sigma_{\epsilon}|\text{DATA})$

$$= \prod_{i=1}^{n} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left[\prod_{j=1}^{m_i} \frac{1}{\sigma_{\epsilon}} \phi_{\text{nor}}(\zeta_{ij}) \right] f_{\boldsymbol{\beta}}(\beta_{1i}, \ldots, \beta_{ki}; \boldsymbol{\mu}_{\boldsymbol{\beta}}, \Sigma_{\boldsymbol{\beta}}) d\beta_{1i}, \ldots, d\beta_{ki}$$

where $\zeta_{ij} = [y_{ij} - \mathcal{D}(t_{ij}, \beta_{1i}, \ldots, \beta_{ki})]/\sigma_{\epsilon}$ and $f_{\boldsymbol{\beta}}(\beta_{1i}, \ldots, \beta_{ki}; \boldsymbol{\mu}_{\boldsymbol{\beta}}, \Sigma_{\boldsymbol{\beta}})$ is the multivariate normal distribution density function.

- Each evaluation of $L(\boldsymbol{\mu}_{\boldsymbol{\beta}}, \Sigma_{\boldsymbol{\beta}}, \sigma_{\epsilon}|\text{DATA})$ will, in general, require numerical approximation of n integrals of dimension k.

- Maximization of $L(\boldsymbol{\mu}_{\boldsymbol{\beta}}, \Sigma_{\boldsymbol{\beta}}, \sigma_{\epsilon}|\text{DATA})$ computationally difficult.

- We use the Pinheiro and Bates (1995b) S-PLUS software for the Lindstrom and Bates (1990) approximate ML to do the fitting.

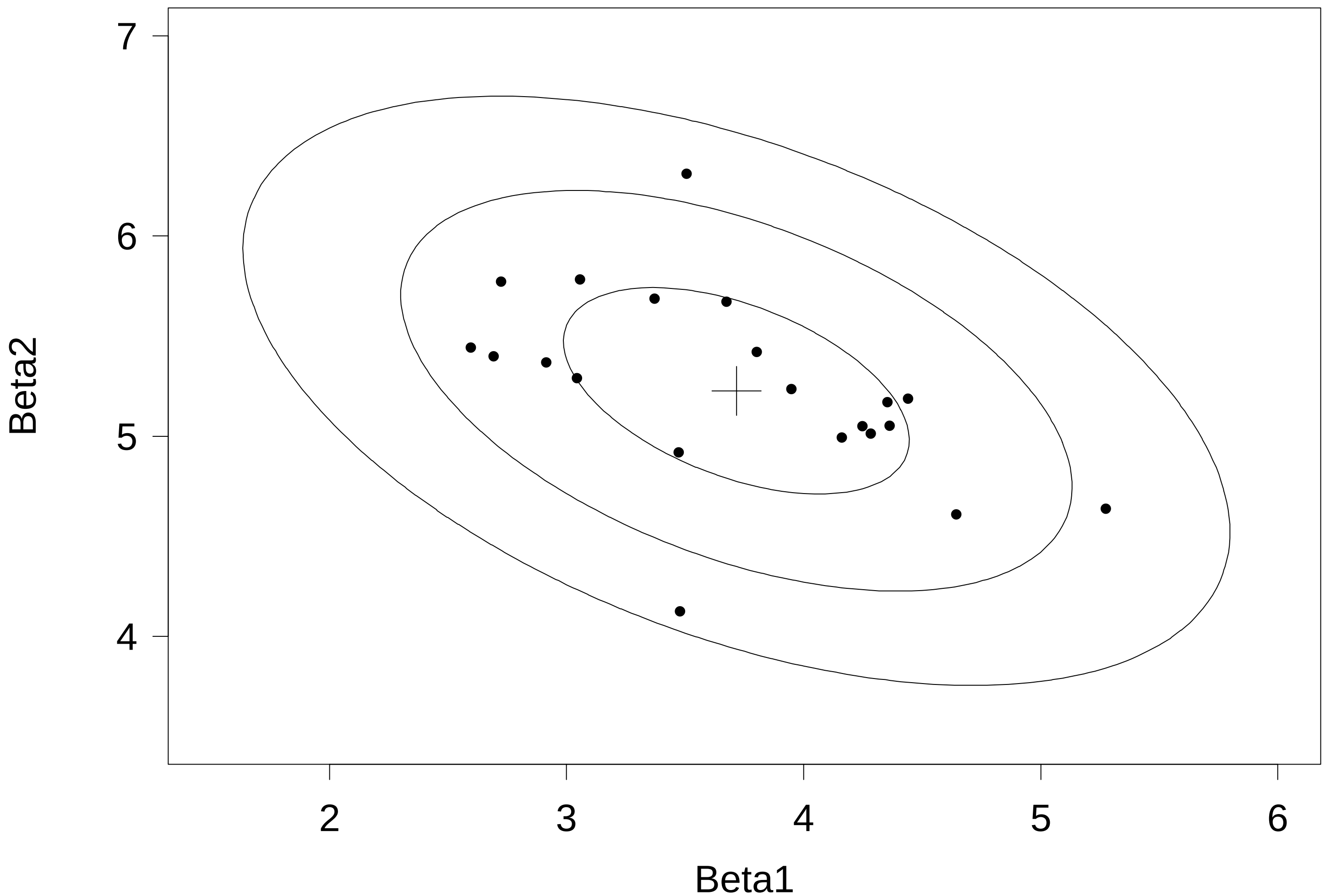
Estimates of Fatigue Data Model Parameters for Alloy-A

The program of Pinheiro and Bates (1995b) gives the following approximate ML estimates.

$$\widehat{\boldsymbol{\mu}}_{\boldsymbol{\beta}} = \begin{pmatrix} 5.17 \\ 3.73 \end{pmatrix}, \qquad \widehat{\boldsymbol{\Sigma}}_{\boldsymbol{\beta}} = \begin{pmatrix} .251 & -.194 \\ -.194 & .519 \end{pmatrix}$$

and $\widehat{\sigma}_{\epsilon} = .0034$.

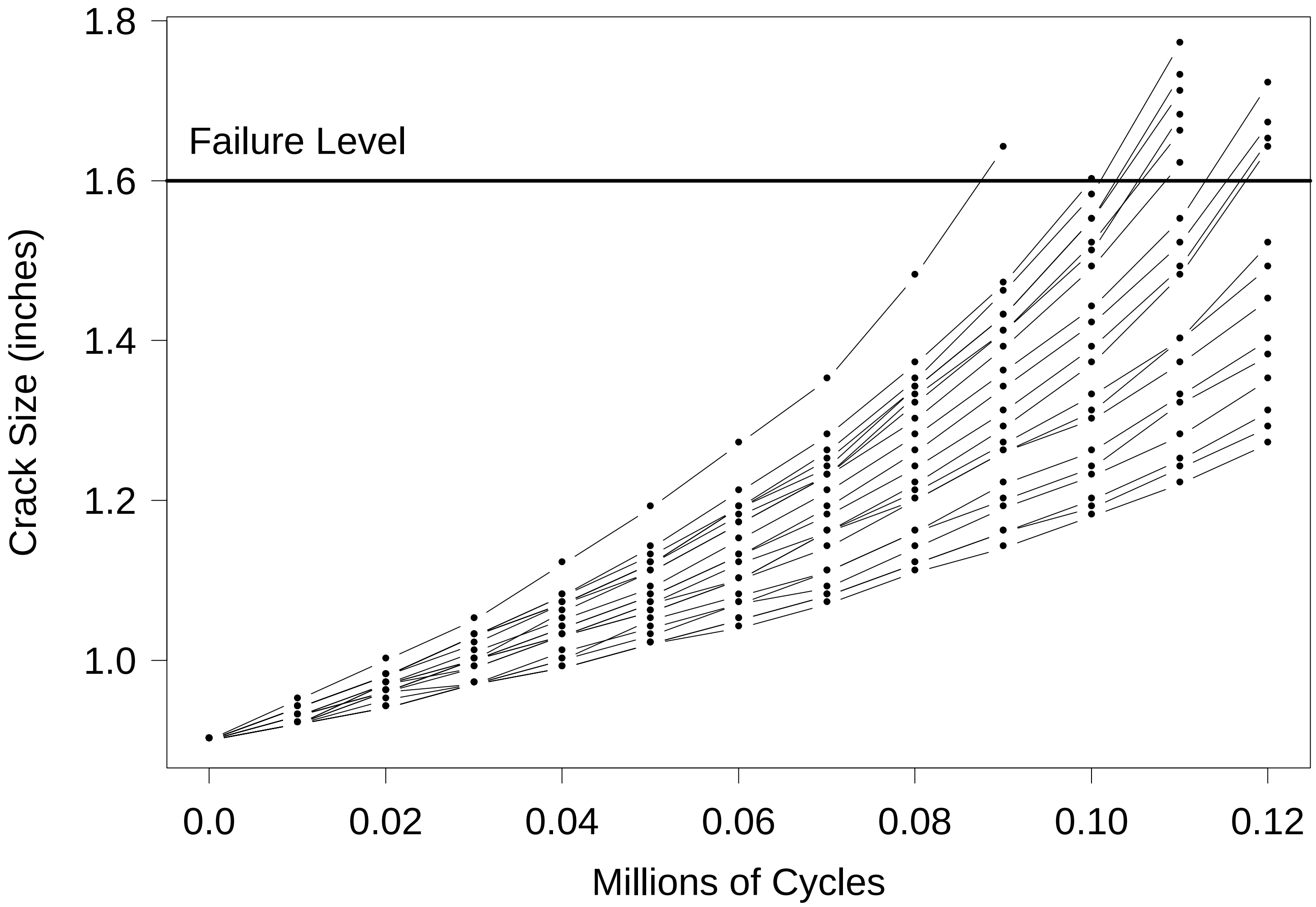
Estimates $\widehat{\beta}_{1i}$ Versus $\widehat{\beta}_{2i}$, $i = 1, \ldots, 21$ and Contours for the Fitted Bivariate Normal Distribution



Models Relating Degradation and Failure

- **Soft failures: specified degradation level**
  - ▶ In some products there is a gradual loss of performance (e.g., decreasing light output from a fluorescent light bulb).
  - ▶ Use fixed $\mathcal{D}_{\mathsf{f}}$ to denote the critical level for the degradation path.
- **Hard failures: correlation between failure and degradation level**
  - ▶ Loss of functionality.
  - ▶ Random $\mathcal{D}_{\mathsf{f}}$. Use a joint distribution of $\mathcal{D}_{\mathsf{f}}$ and other random parameters.

Fatigue Crack Size Observations for Alloy-A



Evaluation of $F(t)$

- **Direct evaluation of $F(t)$:** Closed forms available for simple problems (e.g., a single random variable and other special cases).

- **Numerical integration:** Useful for a small number of random variables (e.g., 2 or 3).

- **FORM (first order) approximation:** Rapid computation, but uncertain approximation.

- **Monte Carlo simulation:** General method. Needs much computer time to evaluate small probabilities. Can use **importance sampling**.

Estimate failure probabilities by evaluating at ML estimates.

Evaluation of $F(t)$ by Numerical Integration

- The failure (crossing probability) can be expressed as

$$\mathrm{Pr}(T \leq t) = F(t) = F(t; \boldsymbol{\theta}_{\boldsymbol{\beta}}) = \mathrm{Pr}[\mathcal{D}(t, \beta_1, \ldots, \beta_k) > \mathcal{D}_{\mathsf{f}}].$$

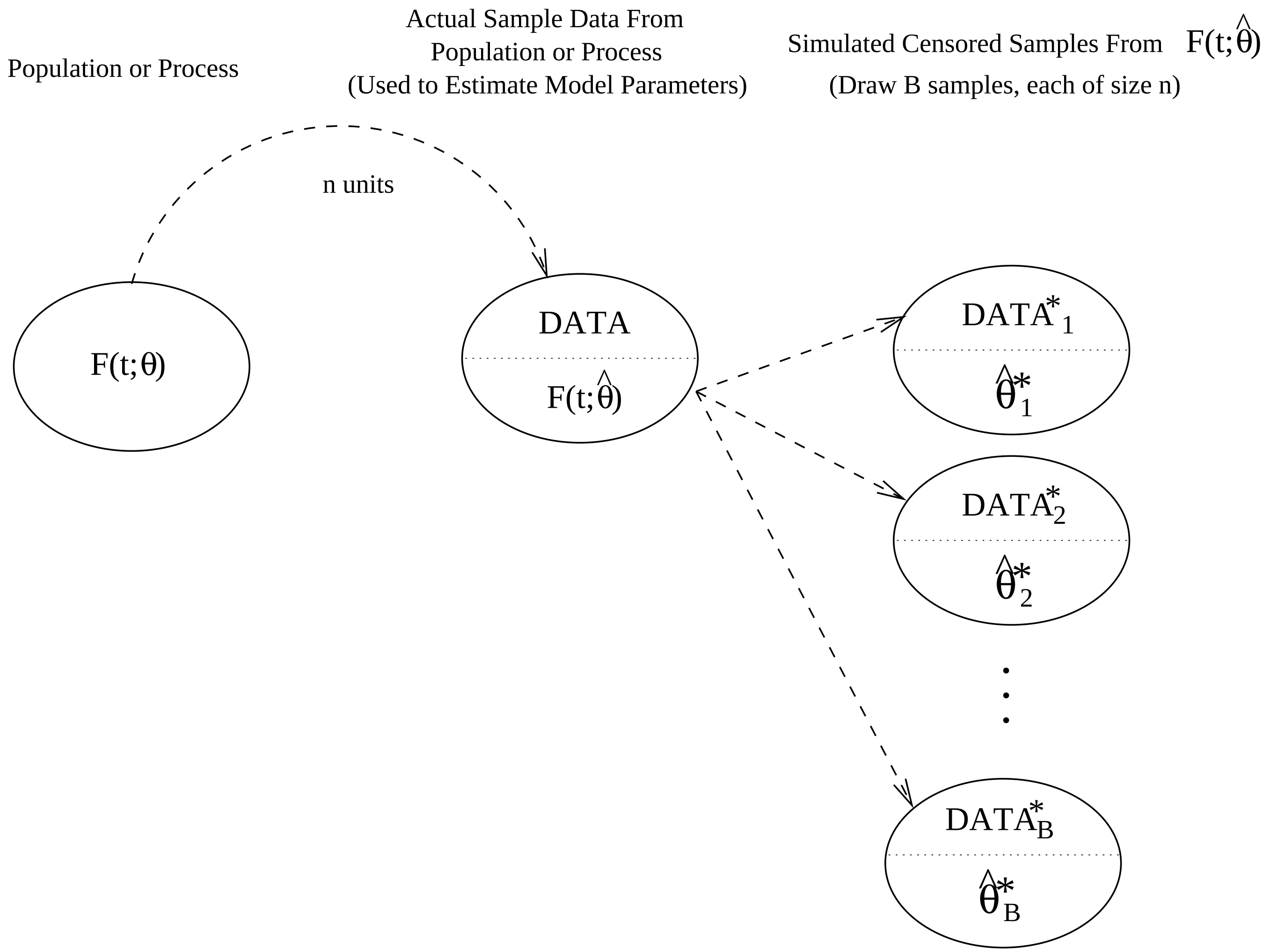
- If (β_1, β_2) follows a bivariate normal distribution with parameters $\mu_{\beta_1}, \mu_{\beta_2}\sigma^2_{\beta_1}, \sigma^2_{\beta_2}, \rho_{\beta_1,\beta_2}$, then $P(T \leq t)$

$$= \int_{-\infty}^{\infty} \Phi_{\mathrm{nor}} \left[- \frac{g(\mathcal{D}_{\mathsf{f}}, t, \beta_1) - \mu_{\beta_2|\beta_1}}{\sigma_{\beta_2|\beta_1}} \right] \frac{1}{\sigma_{\beta_1}} \phi_{\mathrm{nor}} \left(\frac{\beta_1 - \mu_{\beta_1}}{\sigma_{\beta_1}} \right) d\beta_1$$

where $g(\mathcal{D}_{\mathsf{f}}, t, \beta_1)$ is the value of β_2 for given β_1, that gives $\mathcal{D}(t) = \mathcal{D}_{\mathsf{f}}$.

- Method generalizes to multivariate normal.

A Simple Parametric Bootstrap Sampling Method



Confidence Intervals Based on Bootstrap Sampling

- Simulate new sets of n censored sample paths, using the ML estimates as if they were the true model. Repeat B times.
- Compute $\widehat{F}^*(t)_1, \ldots, \widehat{F}^*(t)_B$, bootstrap estimates of $F(t)$.
- Sort the B values $\widehat{F}^*(t)_1, \ldots, \widehat{F}^*(t)_B$ in increasing order giving $\widehat{F}^*(t)_{[b]}, b = 1, \ldots, B$.
- Lower and upper bounds of pointwise $100(1-\alpha)\%$ confidence intervals for the distribution function $F(t)$ are

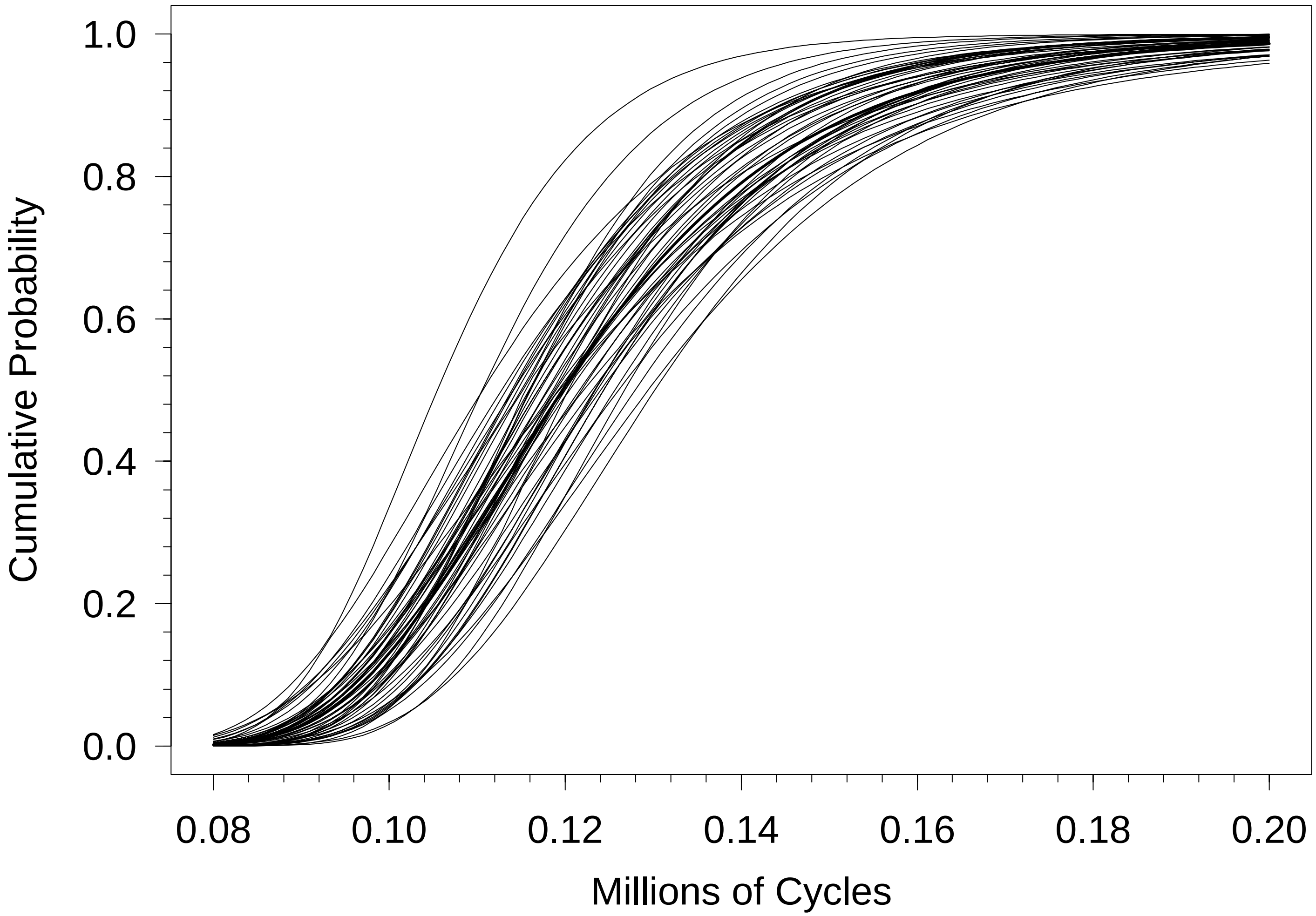
$$\left[\underset{\sim}{F}(t), \quad \widetilde{F}(t)\right] = \left[\widehat{F}^*(t)_{[l]}, \quad \widehat{F}^*(t)_{[u]}\right]$$

where

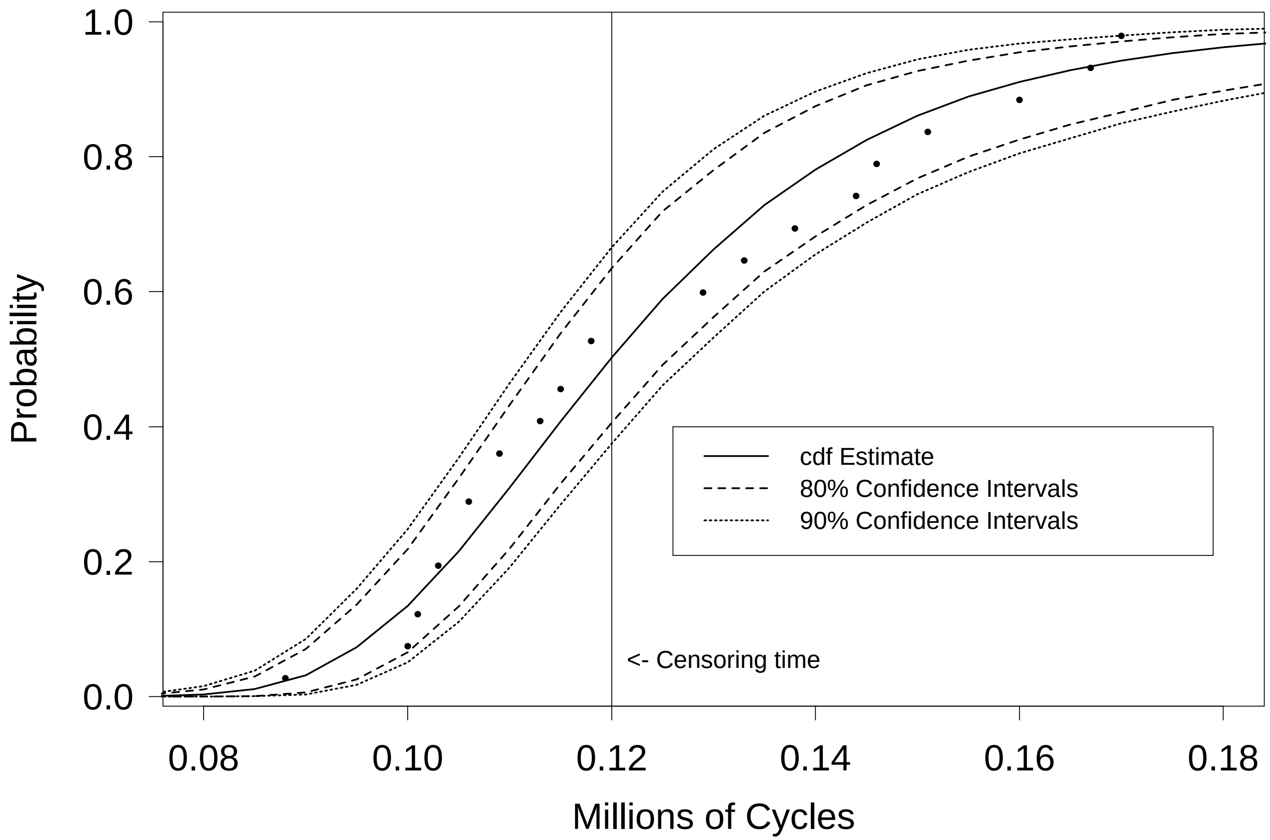
$$l = \Phi_{\text{nor}}\left[2\Phi_{\text{nor}}^{-1}(q) + \Phi_{\text{nor}}^{-1}(\alpha/2)\right], \quad u = \Phi_{\text{nor}}\left[2\Phi_{\text{nor}}^{-1}(q) + \Phi_{\text{nor}}^{-1}(1-\alpha/2)\right],$$

and q is the proportion of the B values of $\widehat{F}^*(t)$ that are less than $\widehat{F}(t)$ (using $q = .5$ is equivalent to the percentile bootstrap method).

Bootstrap Estimates of $F(t)$



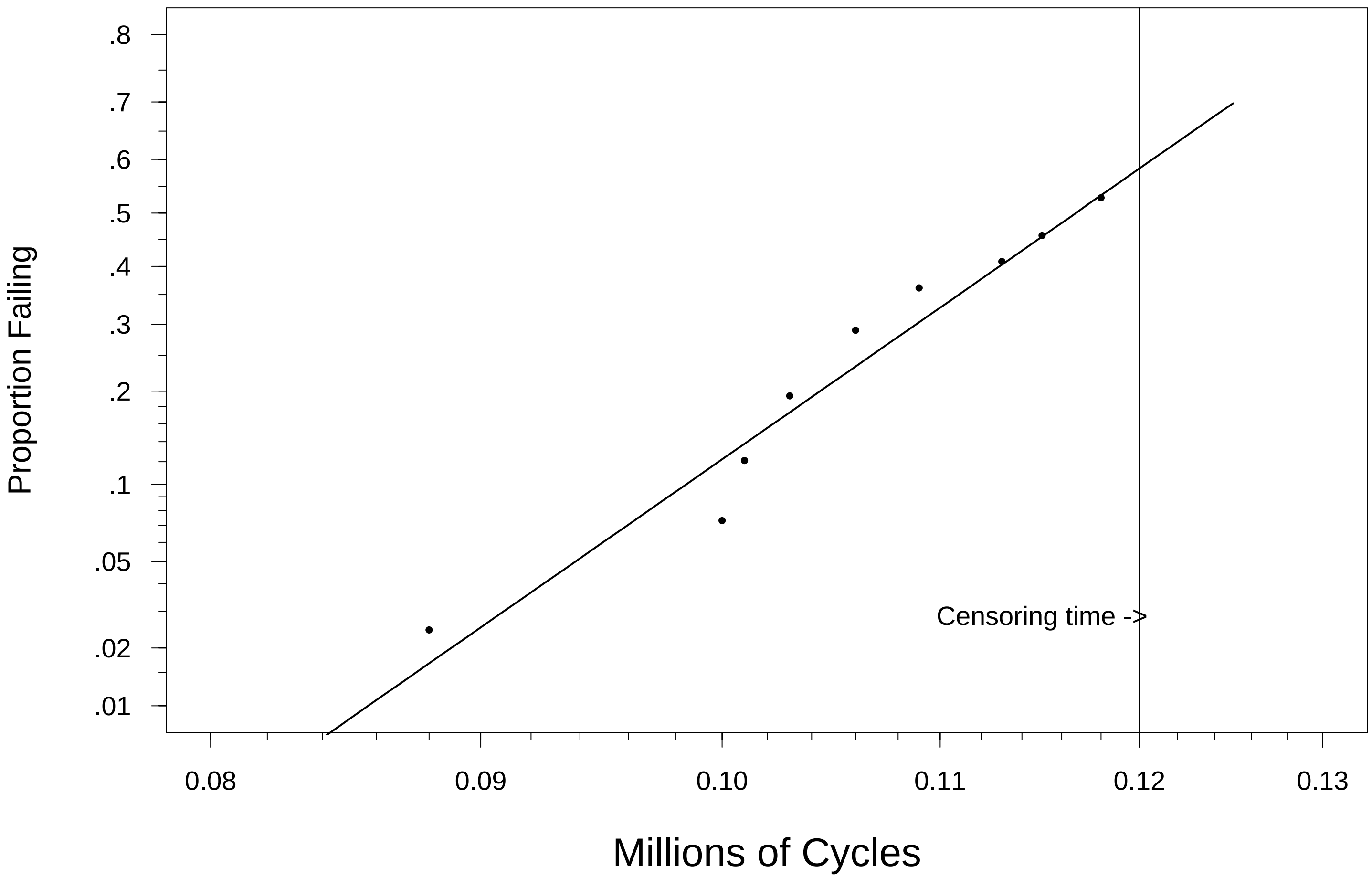
Degradation Estimate of $F(t)$ with Pointwise Two-Sided 90% and 80% Bootstrap Bias-Corrected Percentile Confidence Intervals, Based on the Crack-Size Data Censored at $t_c = .12$. The Nonparametric Estimate of $F(t)$ Indicated by Dots



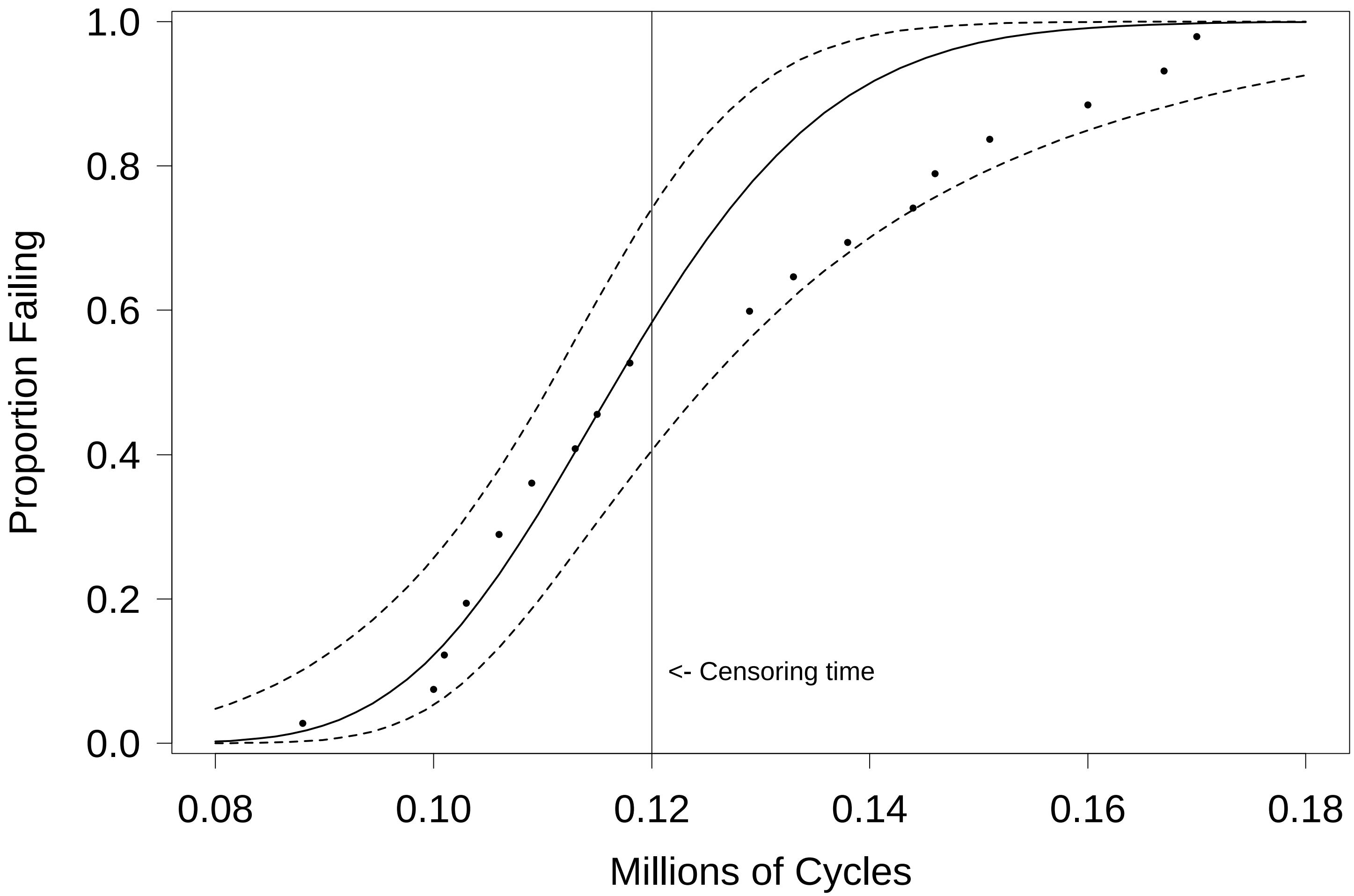
Comparison with Traditional Failure Time Analyses

- Lognormal distribution provides a good fit to the failure-time data up to $t_c = .12$, but not beyond.
- Other commonly used parametric models, which fit almost as well before $t_c = .12$, do not do any better beyond $t_c = .12$.
- The degradation analysis provides a reasonable extrapolation beyond $t_c = .12$—uses information in censored observations more effectively.
- Confidence intervals based on the degradation and failure-time data have similar widths from $.10 < t < .12$. Degradation analysis has tighter bounds for $t > .12$.
- Degradation method provides a much tighter upper confidence bound on the cdf in the lower tail of the failure-time distribution.

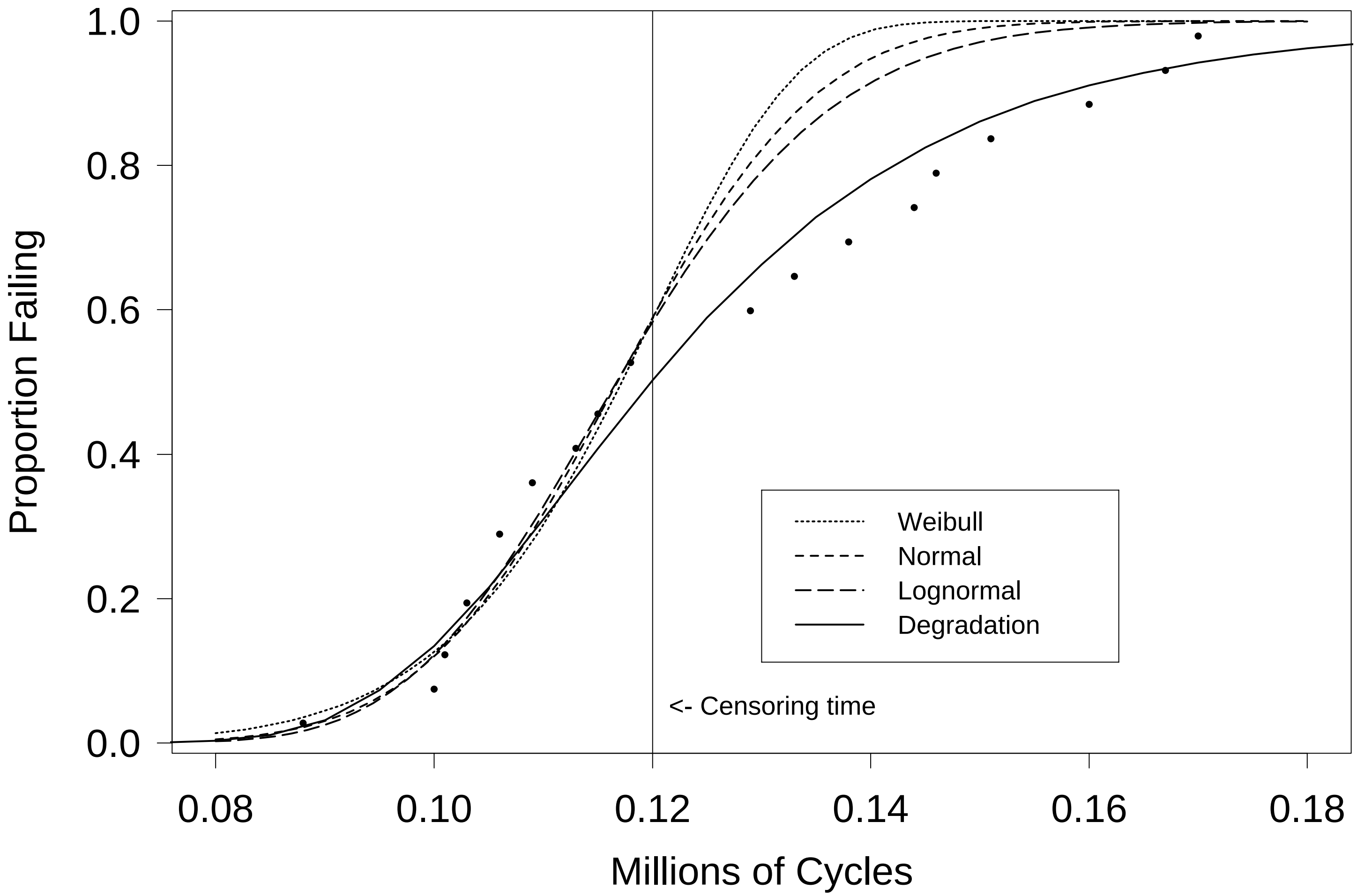
Lognormal Probability Plot and Lognormal Distribution ML Estimate Based on the Failure-Time Data Censored at $t_c = .12$



Lognormal Distribution ML Estimate, Pointwise 90% Approximate Confidence Intervals, and Nonparametric Estimate Based on the Censored Failure-Time Data



Degradation and Failure-Time Data Analysis Based on Data Censored at $t_c = .12$ Million Cycles, Compared with the Nonparametric Estimate



Approximate Degradation Analysis

- An alternative (but only **approximately** correct) method of analyzing degradation data is as follows.

- Do a separate analysis for each unit to predict the time at which the unit will reach the critical degradation level corresponding to failure.

- These predicted failure times are called **pseudo** failure times.

- The n **pseudo** failure times are analyzed as a complete sample of failure times to estimate $F(t)$.

Approximate Degradation Analysis–Details

In detail, the method is as follows:

- For the unit i, use the sample path data (t_{i1}, y_{i1}), $\ldots$, (t_{im_i}, y_{im_i}) and the path model

$$y_{ij} = \mathcal{D}_{ij} + \epsilon_{ij}$$

  to find the (conditional) ML estimate of $\boldsymbol{\beta}_i = (\beta_{1i}, \ldots, \beta_{ki})$, say $\widehat{\boldsymbol{\beta}}_i$. This can be done using nonlinear least squares.

- Solve the equation $\mathcal{D}(t, \widehat{\boldsymbol{\beta}}_i) = \mathcal{D}_{\mathsf{f}}$ for t and call the solution $\widehat{t}_i$.

- Repeat the procedure for each sample path to obtain the pseudo failure time $\widehat{t}_1, \ldots, \widehat{t}_n$.

- Do a single distribution analysis of the data $\widehat{t}_1, \ldots, \widehat{t}_n$ to estimate $F(t)$.

Approximate Degradation Analysis
Simple Linear Path

- For some simple degradation processes

$$\mathcal{D}(t) = \beta_1 + \beta_2 t.$$

- This model is sometime obtained after log transformations on the sample degradation values or on the time scale or both.

- In this case the **pseudo** times to failure are obtained from

$$\widehat{t}_i = \frac{\mathcal{D}_{\mathsf{f}} - \widehat{\beta}_{1i}}{\widehat{\beta}_{2i}}$$

where

$$\widehat{\beta}_{1i} = \bar{y}_i - \widehat{\beta}_{2i} \times \bar{t}_i, \qquad \widehat{\beta}_{2i} = \frac{\sum_{j=1}^{m_i}(t_{ij} - \bar{t}_i) \times y_{ij}}{\sum_{j=1}^{m_i}(t_{ij} - \bar{t}_i)^2}$$

and $\bar{t}_i$ and $\bar{y}_i$ are the means of $t_{i1}, \ldots, t_{im_i}$ and $y_{i1}, \ldots, y_{im_i}$, respectively.

Approximate Degradation Analysis
Simple Linear Path Through the Origin

- For some degradation processes, all paths start at the origin ($t_{i1} = 0, y_{i1} = 0$). If, in addition, the degradation rate is constant, then the degradation path has the form

$$\mathcal{D}(t) = \beta_2 t.$$

- Then the pseudo times to failure are obtained from

$$\widehat{t}_i = \frac{\mathcal{D}_{\mathsf{f}}}{\widehat{\beta}_{2i}}$$

where

$$\widehat{\beta}_{2i} = \frac{\sum_{j=1}^{m_i} t_{ij} \times y_{ij}}{\sum_{j=1}^{m_i} t_{ij}^2}.$$

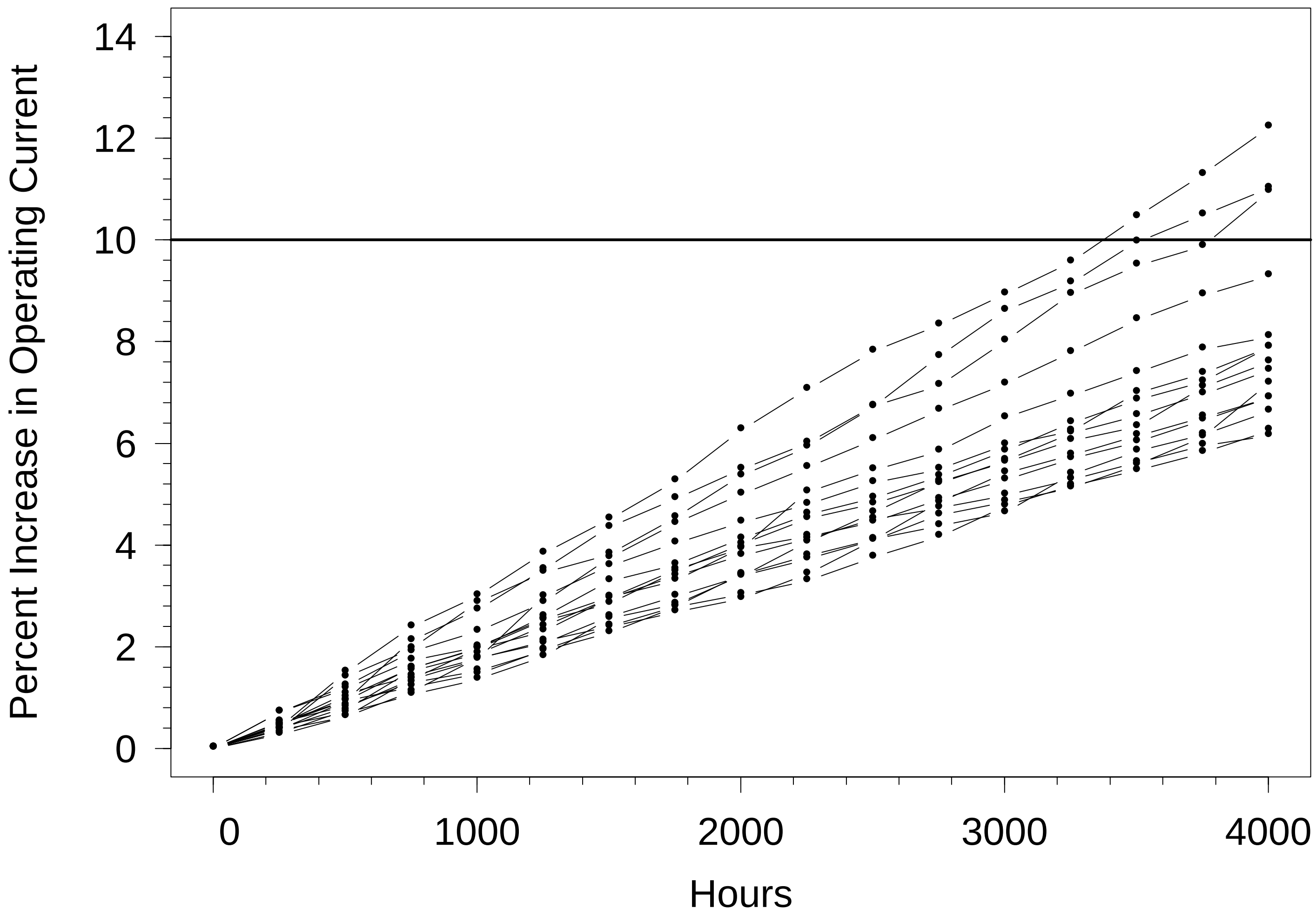
Laser Life Data

- Percentage increase in operating current for GaAs lasers tested at 80°C.

- Fifteen (15) devices each measured every 250 hours up to 4000 hours of operation.

- For these devices and the corresponding application, a $\mathcal{D}_{\mathsf{f}} =$ 10% increase in current was the specified failure level.

Plot of Laser Operating Current as a Function of Time



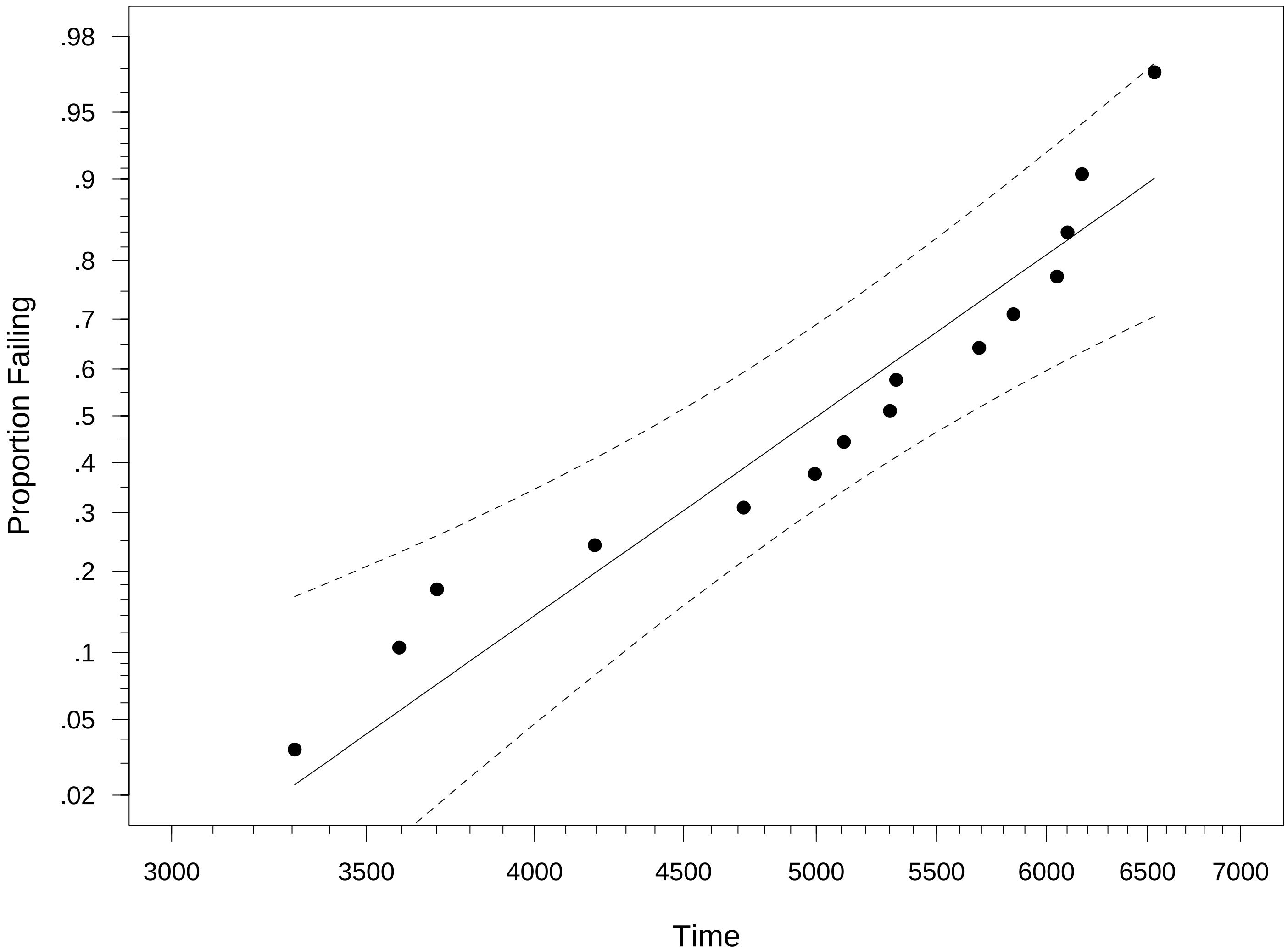
Laser Life Analysis

- The failure times (for paths exceeding $\mathcal{D}_{\mathsf{f}} = 10\%$ increase in current before 4000 hours) and the **pseudo** failure times were obtained by fitting straight lines through the data for each path.

- These pseudo times to failure are 3702, 4194, 5847, 6172, 5301, 3592, 6051, 6538, 5110, 3306, 5326, 4995, 4721, 5689, and 6102 hours.

- One can use methods from Chapters 6 and 8 to obtain the Weibull probability plot of these **pseudo** failure times with the corresponding ML estimate for $F(t)$.

Weibull Probability Plot of the Laser Pseudo Times to Failure Showing the ML Estimate of $F(t)$ and Approximate 95% Pointwise Confidence Intervals



Comments on the Approximate Degradation Analysis

The approximate method may give adequate results if

- The degradation paths are relatively simple.
- The fitted path model is approximately correct.
- There are enough data for precise estimation of the $\boldsymbol{\beta}_i$ parameters for each device.
- The amount of measurement error is small.
- There is not too much extrapolation in predicting the $\widehat{t}_i$ **times to failure**.

Potential Problems With the Approximate Degradation Analysis

- The method ignores the prediction error in $\widehat{t}$ and does not account for measurement error in the observed sample paths.

- The distributions fitted to the **pseudo** times to failure will not, in general, correspond to the distribution induced by the degradation model.

- Some of the sample paths may not contain enough information to estimate all of the path parameters (e.g., when the path model has an asymptote but the sample path has not begun to level off).

  This might necessitate fitting different models for different sample paths in order to predict the crossing time.

Other Topics in Chapter 13

- Autocorrelation in degradation data.
- More details on methods of evaluating $F(t)$.
- More details on ML estimation of degradation parameters.
- More details on the bootstrap method.

Accelerated degradation analysis covered in Chapter 21.