Kinematics and Kinetics of Marine Vessels

(Module 3)

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Kinematics

Description of geometrical aspects of motion without regard to the forces that create the motion.

Kinematics

The objectives of kinematics are

- Define a set of reference frames from which the motion will be described.
- Define sets of coordinate systems on the reference frames, which will be use to express in quantitative way the motion variables.
- Determine relations between acceleration, velocity and position of each point in the system under study.

Reference frames

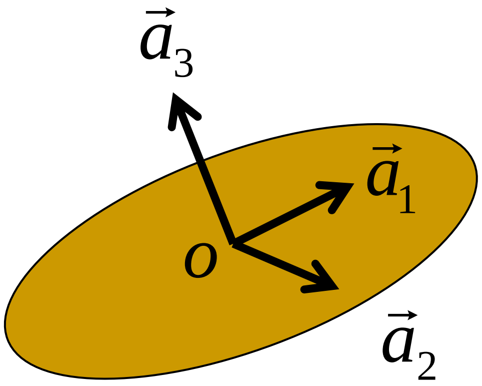
- A reference frame is perspective from which the motion is described.
- Let C be a collection of at least 3 noncolinear points in the Euclidean space. Then C is a reference frame if the distance between every pair of points does not vary with time.
- Then, any rigid body is a reference frame, and any planar rigid object is a reference frame. A point in the space is not a reference frame.

Coordinate systems

- Coordinate systems are fixed to reference frames to express the motion variables.

- A coordinate system consists of a O point fixed in the reference frame and a dextral orthonormal basis, which provides a way to resolve a vector in the space.

- When there is only one coordinate system attached to a reference frame, it is common not to make a distinction between reference frame and coordinate system.



Reference frame

Coordinate system:

$$\{a\} \equiv (o, \vec{a}_1, \vec{a}_2, \vec{a}_3)$$

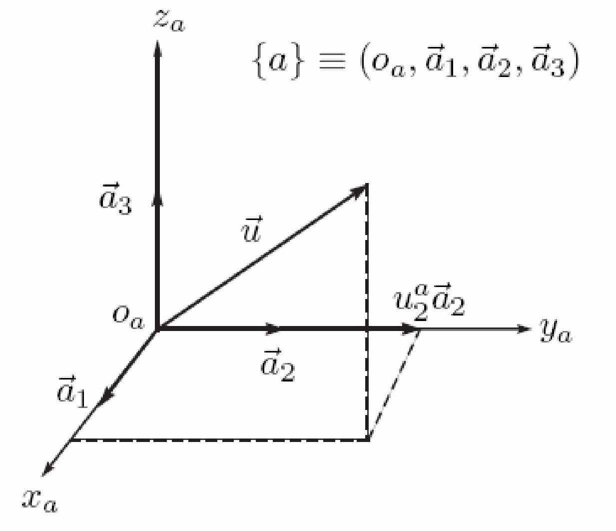
Vector notation

- Variables associated with motion are described by directed line segments or vectors:

Measure along $\vec{a}_1$

$$\vec{u} = \underbrace{u_1^a \vec{a}_1} + \underbrace{u_2^a \vec{a}_2} + \underbrace{u_3^a \vec{a}_3}.$$

Components of $\vec{u}$ in $\{a\}$



Vector notation

- The vectors defined as directed line segments (coordinate free representation) belong a to a three-dimensional space:

$$\vec{u} \in \mathbb{V}^3$$

- This space and the Euclidian space are isomorphic (can be put into a one-to-one correspondence that preserves the their structure)

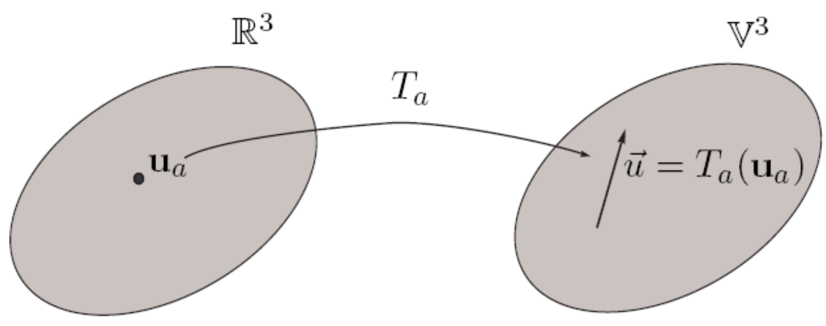
- Exploiting this, it is convenient to consider the coordinate vector representation in a given coordinate system:

$$\mathbf{u}^a \triangleq \begin{bmatrix} u_1^a \\ u_2^a \\ u_3^a \end{bmatrix} = [u_1^a, u_2^a, u_3^a]^T.$$

Coordinate vectors are always given relative to a basis (coordinate system)

Vector notation

$$\mathbf{u}^a \triangleq \begin{bmatrix} u_1^a \\ u_2^a \\ u_3^a \end{bmatrix} \qquad \Leftrightarrow \qquad \vec{u} = T_a(\mathbf{u}^a) = u_1^a \vec{a}_1 + u_2^a \vec{a}_2 + u_3^a \vec{a}_3$$



It is convenient to be familiar with both representations. The coordinate form is convenient for computation and matrix representations. The coordinate-free form allows extending scalar to vector calculus.

Dot and Cross products

Dot product:

$$\vec{u} \cdot \vec{v} = \sum_{j=1}^{3} u_j v_j = \mathbf{u}^T \mathbf{v},$$

Cross product:

$$\vec{c} = \vec{a} \times \vec{b} \quad \Leftrightarrow \quad \mathbf{c} = \mathbf{S}(\mathbf{a})\mathbf{b},$$

Skew-symmetric form of a coordinate vector:

$$\mathbf{S}(\mathbf{a}) \triangleq \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \qquad \mathbf{S}(\mathbf{a}) = -\mathbf{S}^T(\mathbf{a}).$$

$$\vec{c} = \vec{a} \times \vec{b} = -\vec{b} \times \vec{a} \quad \Leftrightarrow \quad \mathbf{c} = \mathbf{S}(\mathbf{a})\mathbf{b} = -\mathbf{S}(\mathbf{b})\mathbf{a} = \mathbf{S}^T(\mathbf{b})\mathbf{a}.$$

Rotation Matrices

Rotation matrix from $\{a\}$ to $\{b\}$: *The rotation matrix from a coordinate system $\{a\}$ to a coordinate system $\{b\}$ is given by $\mathbf{R}_b^a = [R_{jk}]$ with $R_{jk} = \vec{a}_j \cdot \vec{b}_k$. That is,*

$$\mathbf{R}_b^a \triangleq \begin{bmatrix} (\vec{a}_1 \cdot \vec{b}_1) & (\vec{a}_1 \cdot \vec{b}_2) & (\vec{a}_1 \cdot \vec{b}_3) \\ (\vec{a}_2 \cdot \vec{b}_1) & (\vec{a}_2 \cdot \vec{b}_2) & (\vec{a}_2 \cdot \vec{b}_3) \\ (\vec{a}_3 \cdot \vec{b}_1) & (\vec{a}_3 \cdot \vec{b}_2) & (\vec{a}_3 \cdot \vec{b}_3) \end{bmatrix}, \tag{2.11}$$

where $\vec{a}_j$ and $\vec{b}_k$ are the unit vectors defining $\{a\}$ and $\{b\}$ respectively .

$$\mathbf{R}_b^a \in SO(3) : \{\mathbf{R} \in \mathbb{R}^{3\times 3} | \mathbf{R}\mathbf{R}^T = \mathbf{I}_{3\times 3}; \det(\mathbf{R}) = 1\}.$$

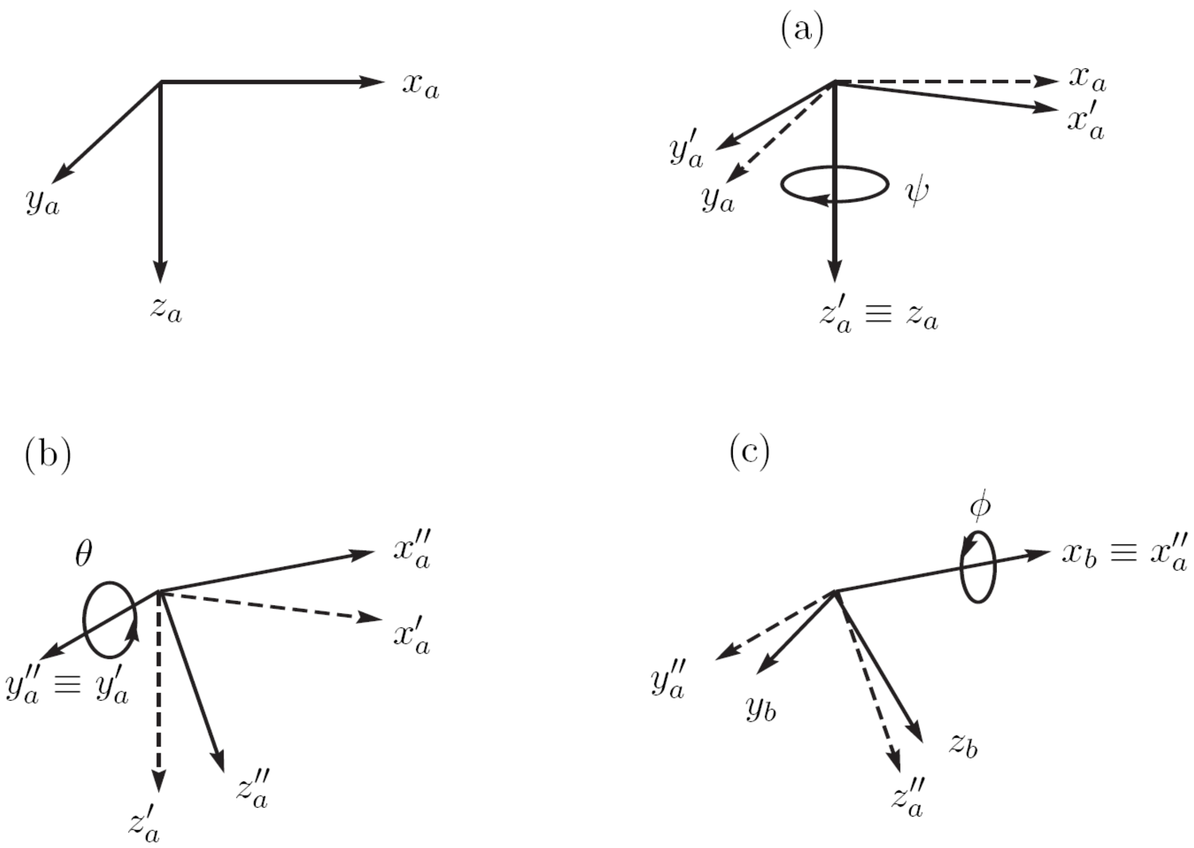
$$(\mathbf{R}_b^a)^{-1} = (\mathbf{R}_b^a)^T = \mathbf{R}_a^b.$$

Euler Angles

- The attitude of one coordinate system relative to another can be described by three consecutive rotations.
- There are 12 ways of doing this depending on the order of the rotations, and each triplet of rotated angles is called a set of Euler Angles

Roll, Pitch and Yaw

The roll, pitch and yaw is set of Euler angles commonly use in guidance and navigation.



Vector of Roll, Pitch and Yaw that take {a} into the orientation of {b}:

$$\mathbf{\Theta}_{ab} \triangleq [\phi, \theta, \psi]^T.$$

Rotation matrix in terms of RPY

$$\mathbf{R}_b^a = \mathbf{R}_{z,\psi}\mathbf{R}_{y',\theta}\mathbf{R}_{x'',\phi}.$$

After multiplication:

$$\mathbf{R}_b^a = \begin{bmatrix} c\psi c\theta & -s\psi c\phi + c\psi s\theta s\phi & s\psi s\phi + c\psi c\phi s\theta \\ s\psi c\theta & c\psi c\phi + s\phi s\theta s\psi & -c\psi s\phi + s\psi c\phi s\theta \\ -s\theta & c\theta s\phi & c\theta c\phi \end{bmatrix}$$

$s \equiv \sin(\cdot)$ $\qquad$ $c \equiv \cos(\cdot)$

Note that the multiplication is consistent with the transformation

$$\mathbf{r}^a = \mathbf{R}_b^a\,\mathbf{r}^b$$

Transforms a vector from the base b to the base a.

Angular velocity

Since the rotation matrix is orthogonal, then

$$\frac{d}{dt}[\mathbf{R}_b^a(\mathbf{R}_b^a)^T] = \dot{\mathbf{R}}_b^a(\mathbf{R}_b^a)^T + \mathbf{R}_b^a(\dot{\mathbf{R}}_b^a)^T = \mathbf{0}$$

This imply that $\dot{\mathbf{R}}_b^a(\mathbf{R}_b^a)^T$ is skew symmetric, and hence be represented by a single coordinate vector (Egeland and Gravdahl, 2002):

$$\boldsymbol{\omega}_{ab}^a : \qquad \mathbf{S}(\boldsymbol{\omega}_{ab}^a) = \dot{\mathbf{R}}_b^a(\mathbf{R}_b^a)^T$$

This is the angular velocity of {b} with respect to {a}, expressed in {a}

The derivative of the rotation matrix can then be expressed as

$$\dot{\mathbf{R}}_b^a = \mathbf{S}(\boldsymbol{\omega}_{ab}^a)\,\mathbf{R}_b^a = \mathbf{R}_b^a\,\mathbf{S}(\boldsymbol{\omega}_{ab}^b).$$

Angular velocity and deriv of RPY

Consider a rotation from {a} to {d} via RPY:

$$\mathbf{R}_b^a = \mathbf{R}_{z,\psi}, \qquad \mathbf{R}_c^b = \mathbf{R}_{y,\theta}, \qquad \mathbf{R}_d^c = \mathbf{R}_{x,\phi}$$

The angular velocities are

$$\begin{aligned}\boldsymbol{\omega}_{ab}^a &= [0, 0, \dot{\psi}]^T, \\ \boldsymbol{\omega}_{bc}^b &= [0, \dot{\theta}, 0]^T, \\ \boldsymbol{\omega}_{cd}^c &= [\dot{\phi}, 0, 0]^T.\end{aligned}$$

Then from the theorem of addition of angular velocities we have

$$\boldsymbol{\omega}_{ad}^a = \boldsymbol{\omega}_{ab}^a + \mathbf{R}_b^a \boldsymbol{\omega}_{bc}^b + \mathbf{R}_b^a \mathbf{R}_c^b \boldsymbol{\omega}_{cd}^c$$

$$\boldsymbol{\omega}_{ad}^d = \mathbf{R}_a^d \boldsymbol{\omega}_{ab}^a + \mathbf{R}_c^d \mathbf{R}_b^c \boldsymbol{\omega}_{bc}^b + \mathbf{R}_c^c \boldsymbol{\omega}_{cd}^c$$

Angular velocity and deriv of RPY

From

$$\boldsymbol{\omega}_{ad}^{a} = \boldsymbol{\omega}_{ab}^{a} + \mathbf{R}_b^a \boldsymbol{\omega}_{bc}^{b} + \mathbf{R}_b^a \mathbf{R}_c^b \boldsymbol{\omega}_{cd}^{c}$$

$$\boldsymbol{\omega}_{ad}^{d} = \mathbf{R}_a^d \boldsymbol{\omega}_{ab}^{a} + \mathbf{R}_c^d \mathbf{R}_b^c \boldsymbol{\omega}_{bc}^{b} + \mathbf{R}_c^c \boldsymbol{\omega}_{cd}^{c}$$

using,

$$\boldsymbol{\Theta}_{ad} \triangleq [\phi, \theta, \psi]^T$$

we obtain

$$\boldsymbol{\omega}_{ad}^{a} = \mathbf{E}_a(\boldsymbol{\Theta}_{ad})\dot{\boldsymbol{\Theta}}_{ad} = \begin{bmatrix} c\psi c\theta & -s\psi & 0 \\ s\psi c\theta & c\psi & 0 \\ -s\theta & 0 & 1 \end{bmatrix} \dot{\boldsymbol{\Theta}}_{ad}$$

$$\boldsymbol{\omega}_{ad}^{d} = \mathbf{E}_d(\boldsymbol{\Theta}_{ad})\dot{\boldsymbol{\Theta}}_{ad} = \begin{bmatrix} 1 & 0 & -s\theta \\ 0 & c\phi & s\phi c\theta \\ 0 & -s\phi & c\phi c\theta \end{bmatrix} \dot{\boldsymbol{\Theta}}_{ad}$$

$$\Leftrightarrow$$

$$\dot{\boldsymbol{\Theta}}_{ad} = \mathbf{E}_a^{-1}(\boldsymbol{\Theta}_{ad})\boldsymbol{\omega}_{ad}^{a} = \begin{bmatrix} \frac{c\psi}{c\theta} & \frac{s\psi}{c\theta} & 0 \\ -s\psi & c\psi & 0 \\ c\psi t\theta & s\psi t\theta & 1 \end{bmatrix} \boldsymbol{\omega}_{ad}^{a}$$

$$\dot{\boldsymbol{\Theta}}_{ad} = \mathbf{E}_d^{-1}(\boldsymbol{\Theta}_{ad})\boldsymbol{\omega}_{ad}^{d} = \begin{bmatrix} 1 & s\phi t\theta & c\phi t\theta \\ 0 & c\phi & -s\phi \\ 0 & \frac{s\phi}{c\theta} & \frac{c\phi}{c\theta} \end{bmatrix} \boldsymbol{\omega}_{ad}^{d}$$

The angular velocity and the derivative of the Euler angles are, in general, different things.

Position

Position of "p" with respect to {a}, and expressed in {a}:

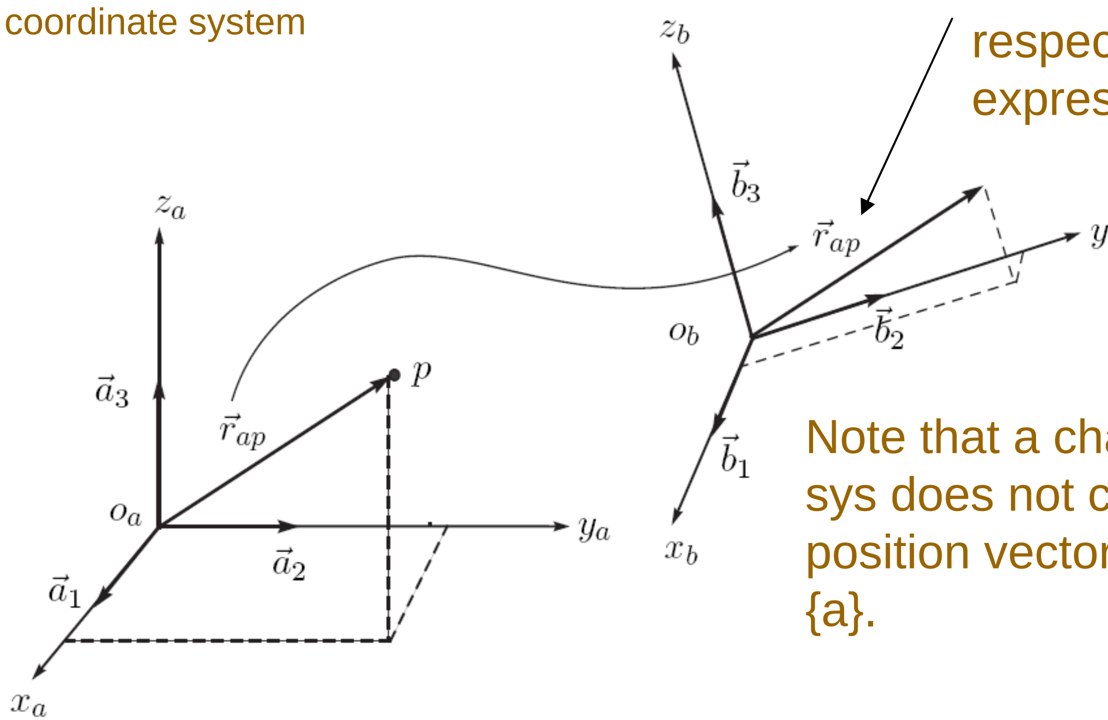
Coordinate system where the vector is expressed

$$\mathbf{r}^a_{ap} = [r^a_{ap,1}, r^a_{ap,2}, r^a_{ap,3}]^T$$

Point of interest

Reference coordinate system

Position of "p" with respect to {a}, and expressed in {b}.

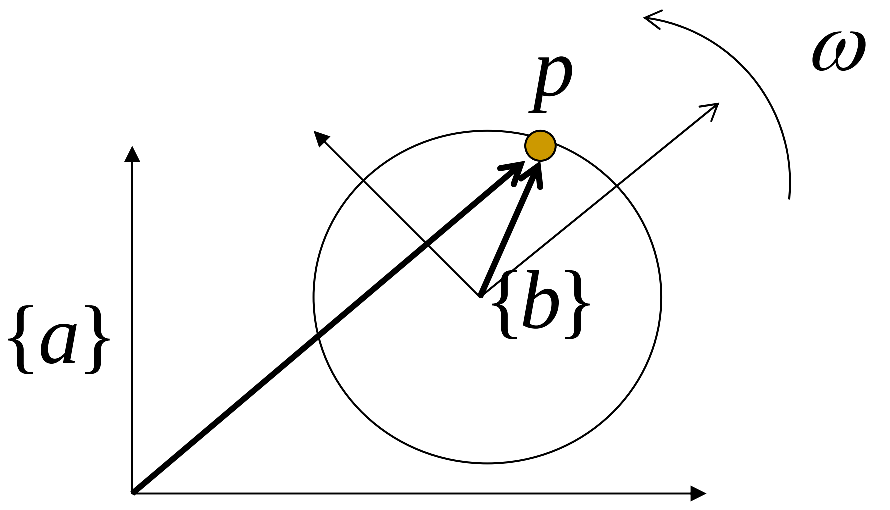


Note that a change in coord sys does not change the position vector; it is still p in {a}.

Velocity

- The relative position of any two points is invariant in any coordinate system (||r1-r2|| is independent of the coordinate system used to express r1 and r2).

- The velocity, however, depends on the coordinate system adopted.



{b} and “p” rotate with angular velocity ω.

The velocity of “p” wrt {b} is zero, but not wrt {a}.

Derivative of a vector

The derivative of a vector makes no sense without specifying the coordinate system with respect of which the derivative is taken:

$$\frac{{}^a d}{dt}\vec{r} \triangleq \frac{dr_1^a}{dt}\vec{a}_1 + \frac{dr_2^a}{dt}\vec{a}_2 + \frac{dr_3^a}{dt}\vec{a}_3,$$

$$\frac{{}^b d}{dt}\vec{r} \triangleq \frac{dr_1^b}{dt}\vec{b}_1 + \frac{dr_2^b}{dt}\vec{b}_2 + \frac{dr_3^b}{dt}\vec{b}_3.$$

In general

$$\frac{{}^a d}{dt}\vec{r} \neq \frac{{}^b d}{dt}\vec{r}.$$

Transport Theorem

$$\frac{{}^a d\vec{r}}{dt} = \frac{{}^b d\vec{r}}{dt} + \vec{\omega}_{ab} \times \vec{r}$$

In coordinate form

$$\mathbf{r}^a = \mathbf{R}_b^a \mathbf{r}^b$$

$$\dot{\mathbf{r}}^a = \mathbf{R}_b^a \dot{\mathbf{r}}^b + \dot{\mathbf{R}}_b^a \mathbf{r}^b$$

$$= \mathbf{R}_b^a [\dot{\mathbf{r}}^b + \mathbf{S}(\boldsymbol{\omega}_{ab}^b) \mathbf{r}^b]$$

If we multiply both sides by $\mathbf{R}_a^b$

$$\dot{\mathbf{r}}^b = \dot{\mathbf{r}}^b + \mathbf{S}(\boldsymbol{\omega}_{ab}^b) \mathbf{r}^b \text{ ?}$$

We need to use double script so we don't have this problem in notation:

$$\dot{\mathbf{r}}_{ap}^b = \dot{\mathbf{r}}_{bp}^b + \mathbf{S}(\boldsymbol{\omega}_{ab}^b) \mathbf{r}_{bp}^b$$

Velocity and Acceleration

If {i} denotes an inertial coordinate system, then

$$\vec{v}_{ip} \triangleq \frac{{}^i d}{dt} \vec{r}_{ip} \qquad \mathbf{v}_{ip}^{i} \triangleq \dot{\mathbf{r}}_{ip}^{i}$$

The first subscript Indicates the coordinate system with respect of which the derivative is taken

Angular acceleration

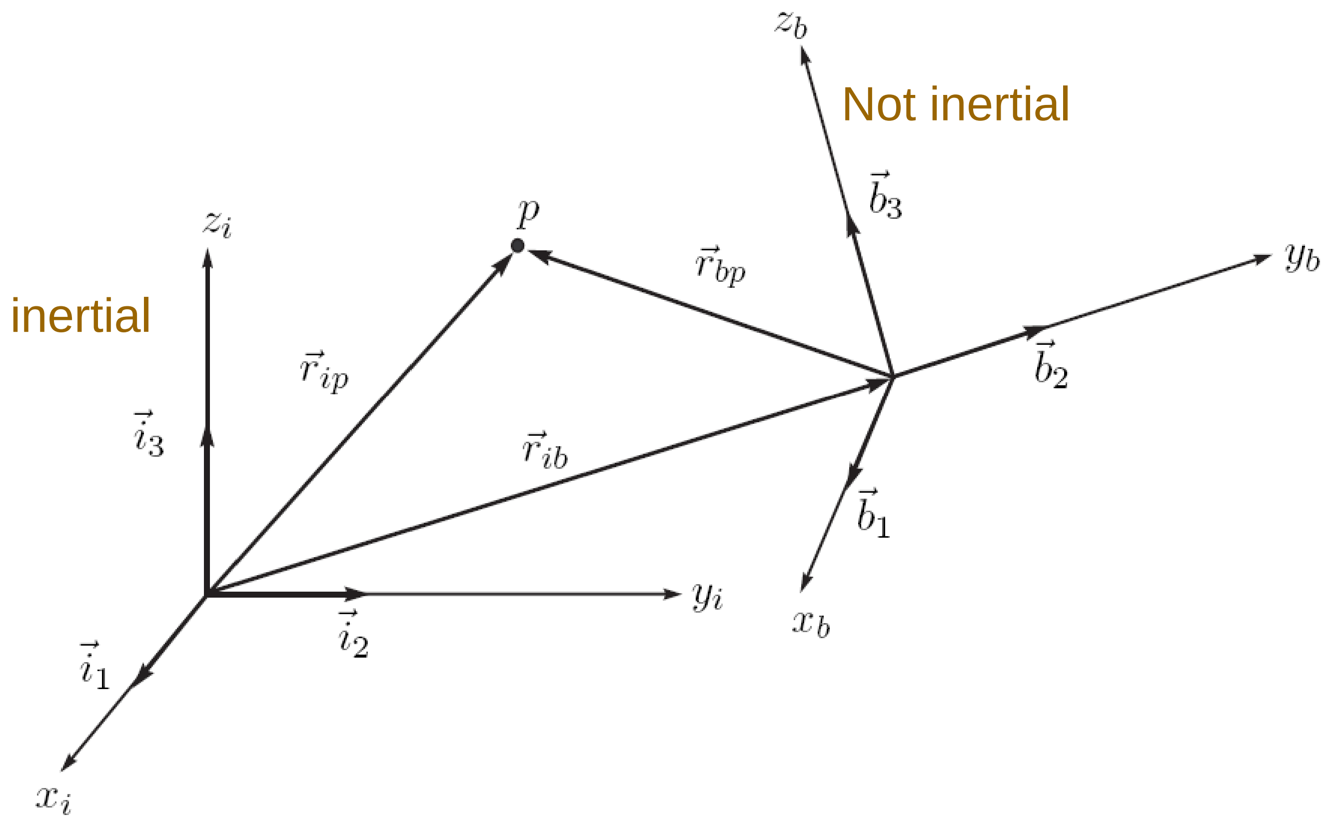
$$\frac{{}^i d}{dt}\vec{\omega}_{ib} = \frac{{}^b d}{dt}\vec{\omega}_{ib} + \underbrace{\vec{\omega}_{ib} \times \vec{\omega}_{ib}}_{=0}.$$

$$\vec{\alpha}_{ib} \triangleq \frac{{}^i d}{dt}\vec{\omega}_{ib} = \frac{{}^b d}{dt}\vec{\omega}_{ib}$$

Linear acceleration:

$$\vec{a}_{ip} \triangleq \frac{{}^i d^2 \vec{r}_{ip}}{dt^2} = \frac{{}^i d\vec{v}_{ip}}{dt}$$

Motion in different coord systems



Motion in different coord systems

$$\vec{r}_{ip} = \vec{r}_{ib} + \vec{r}_{bp}$$

$$\frac{{}^i d}{dt}\vec{r}_{ip} = \frac{{}^i d}{dt}\vec{r}_{ib} + \frac{{}^i d}{dt}\vec{r}_{bp} \quad \Leftrightarrow \quad \frac{{}^i d}{dt}\vec{r}_{ip} = \frac{{}^i d}{dt}\vec{r}_{ib} + \frac{{}^b d}{dt}\vec{r}_{bp} + \vec{\omega}_{ib} \times \vec{r}_{bp}$$

Transport Theorem

In coordinate form

$$\mathbf{v}^i_{ip} = \mathbf{v}^i_{ib} + \mathbf{R}^i_b[\dot{\mathbf{r}}^b_{bp} + \mathbf{S}(\boldsymbol{\omega}^b_{ib})\mathbf{r}^b_{bp}]$$

Motion in different coord systems

$$\frac{{}^i d}{dt}\vec{r}_{ip} = \frac{{}^i d}{dt}\vec{r}_{ib} + \frac{{}^b d}{dt}\vec{r}_{bp} + \vec{\omega}_{ib} \times \vec{r}_{bp}$$

Taking a derivative again:

$$\begin{aligned}
\frac{{}^i d^2}{dt^2}\vec{r}_{ip} &= \frac{{}^i d^2}{dt^2}\vec{r}_{ib} + \frac{{}^i d}{dt}\left(\frac{{}^b d}{dt}\vec{r}_{bp} + \vec{\omega}_{ib} \times \vec{r}_{bp}\right), \\
&= \frac{{}^i d^2}{dt^2}\vec{r}_{ib} + \frac{{}^b d}{dt}\left(\frac{{}^b d}{dt}\vec{r}_{bp} + \vec{\omega}_{ib} \times \vec{r}_{bp}\right) + \vec{\omega}_{ib} \times \left(\frac{{}^b d}{dt}\vec{r}_{bp} + \vec{\omega}_{ib} \times \vec{r}_{bp}\right), \\
&= \frac{{}^i d^2}{dt^2}\vec{r}_{ib} + \frac{{}^b d^2}{dt^2}\vec{r}_{bp} + \frac{{}^b d}{dt}\vec{\omega}_{ib} \times \vec{r}_{bp} + 2\vec{\omega}_{ib} \times \frac{{}^b d}{dt}\vec{r}_{bp} + \vec{\omega}_{ib} \times \vec{\omega}_{ib} \times \vec{r}_{bp}.
\end{aligned}$$

With the adopted notation

$$\vec{a}_{ip} = \vec{a}_{ib} + \frac{{}^b d^2}{dt^2}\vec{r}_{bp} + \underbrace{\vec{\alpha}_{ib} \times \vec{r}_{bp}}_{Transversal} + \underbrace{2\,\vec{\omega}_{ib} \times \frac{{}^b d}{dt}\vec{r}_{bp}}_{Coriollis} + \underbrace{\vec{\omega}_{ib} \times (\vec{\omega}_{ib} \times \vec{r}_{bp})}_{Centripetal}.$$

$$\mathbf{a}_{ip}^i = \mathbf{a}_{ib}^i + \mathbf{R}_b^i[\ddot{\mathbf{r}}_{bp}^b + \mathbf{S}(\boldsymbol{\alpha}_{ib}^b)\mathbf{r}_{bp}^b + 2\,\mathbf{S}(\boldsymbol{\omega}_{ib}^b)\dot{\mathbf{r}}_{bp}^b + \mathbf{S}(\boldsymbol{\omega}_{ib}^b)\mathbf{S}(\boldsymbol{\omega}_{ib}^b)\mathbf{r}_{bp}^b]$$

Ship kinematics

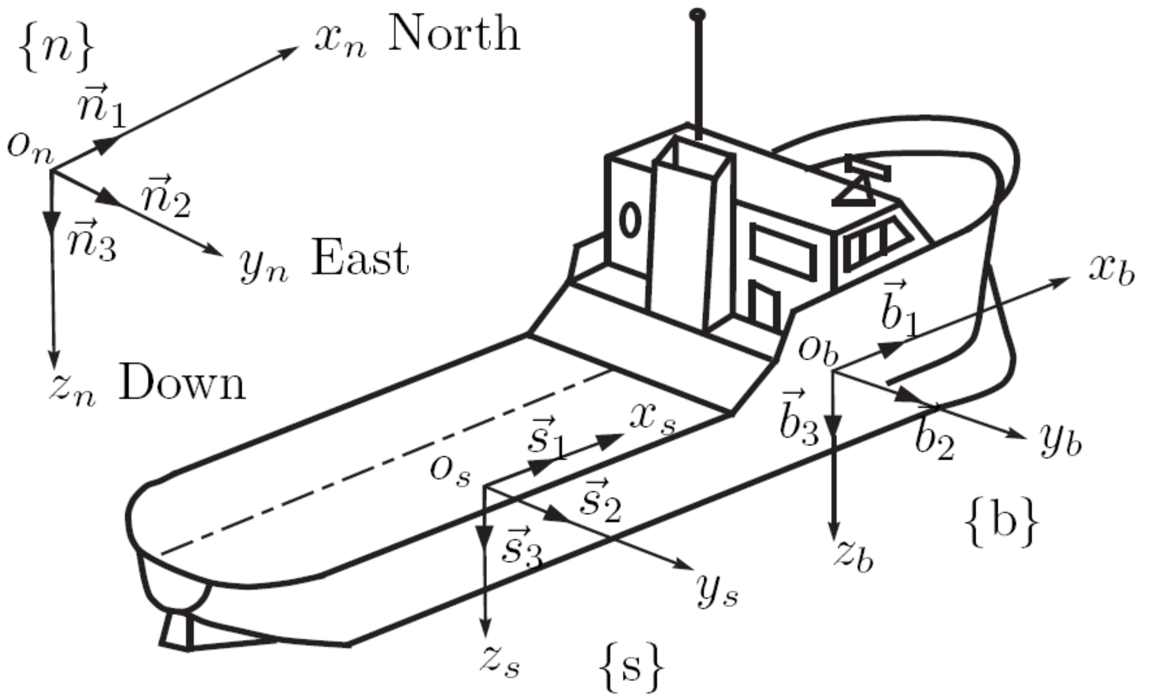
Ship kinematics is different depending on the assumptions made to describe the motion:

- Manoeuvring
- Seakeeping

Ship Motion description

To describe the ship motion the following coordinate systems are used:

- North-East-Down, $\{n\}$;
- Body(-fixed), $\{b\}$;
- Seakeeping, $\{s\}$.

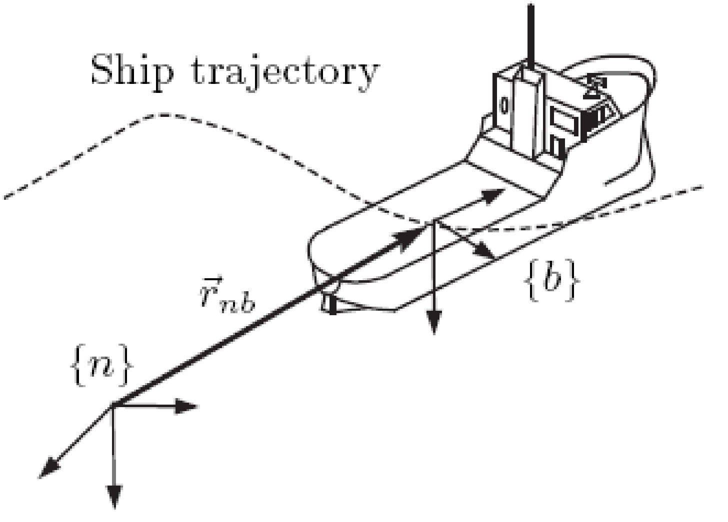


Manoeuvring Kinematics

- In manoeuvring, the position of the vessel is given by the position of of {b}-body-fixed coordinate system with respect to {n}-North-East-Down coordinate system.
- The attitude is given by the angles of roll, pitch and yaw that take {n} into the orientation of {b}.

$$\mathbf{r}^n_{nb} \triangleq [N, E, D]^T$$

$$\boldsymbol{\Theta}_{nb} \triangleq [\phi, \theta, \psi]^T$$



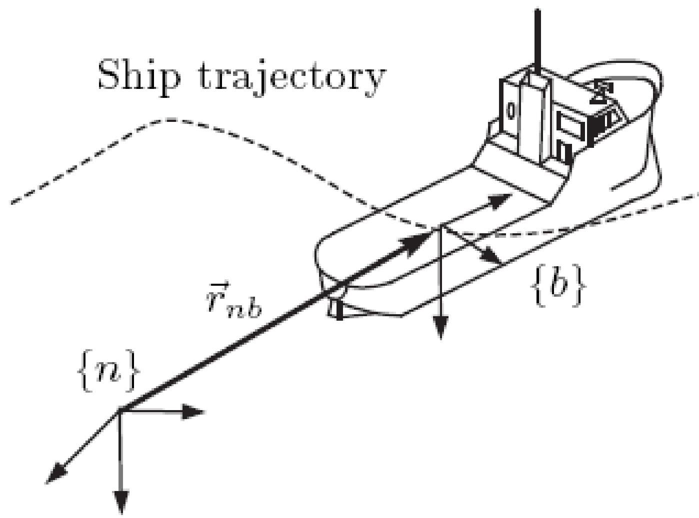
Vessel linear velocities

The velocities are more conveniently expressed in {b}-body-fixed coordinate system:

$$\mathbf{v}_{nb}^{b} \triangleq \mathbf{R}_{n}^{b}\dot{\mathbf{r}}_{nb}^{n}, = \mathbf{R}_{n}^{b}\,[\dot{N}, \dot{E}, \dot{D}]^{T}$$

$$\mathbf{v}_{nb}^{b} = [u, v, w]^{T}$$

surge sway heave



Note that the following integral has no physical meaning:

$$\int_0^t \mathbf{v}_{nb}^{b}\,d\tau$$

Ship trajectory: $\mathbf{r}_{nb}^{n}(t) = \int_0^t \mathbf{R}_{b}^{n}\mathbf{v}_{nb}^{b}\,d\tau + \mathbf{r}_{nb}^{n}(0)$

Vessel angular velocities

The angular velocity expressed in {b}-body-fixed coordinate system is

$$\boldsymbol{\omega}_{nb}^{b} = [p, q, r]^T : \qquad \dot{\mathbf{R}}_b^n = \mathbf{R}_b^n \mathbf{S}(\boldsymbol{\omega}_{nb}^{b})$$

roll pitch yaw

Note that the following integral has no physical meaning:

$$\int_0^t \boldsymbol{\omega}_{nb}^{b}\, d\tau$$

The ship orientation is obtained integrating

$$\dot{\boldsymbol{\Theta}}_{nb} = \mathbf{T}_b(\boldsymbol{\Theta}_{nb})\boldsymbol{\omega}_{nb}^{b}$$

Which is the relationship we have already shown between the ang vel and the derivative of the Euler angles.

Generalised position and velocity

- We define the coordinate position-orientation vector (Fossen, 1994):

$$\boldsymbol{\eta} \triangleq \begin{bmatrix} \mathbf{r}^n_{nb} \\ \boldsymbol{\Theta}_{nb} \end{bmatrix} = [N, E, D, \phi, \theta, \psi]^T$$

- We define the coordinate linear-angular velocity vector (Fossen, 1994):

$$\boldsymbol{\nu} \triangleq \begin{bmatrix} \mathbf{v}^b_{nb} \\ \boldsymbol{\omega}^b_{nb} \end{bmatrix} = [u, v, w, p, q, r]^T$$

Kinematic model {n}-{b}

Then,

$$\dot{\boldsymbol{\eta}} = \mathbf{J}_b^n(\boldsymbol{\eta})\,\boldsymbol{\nu}$$

$$\mathbf{J}_b^n(\boldsymbol{\eta}) \triangleq \begin{bmatrix} \mathbf{R}_b^n(\boldsymbol{\Theta}_{nb}) & \mathbf{0}_{3\times3} \\ \mathbf{0}_{3\times3} & \mathbf{T}_b(\boldsymbol{\Theta}_{nb}) \end{bmatrix}$$

Note that

$$\mathbf{J}_b^n(\boldsymbol{\eta})^{-1} \triangleq \mathbf{J}_n^b(\boldsymbol{\eta}) = \begin{bmatrix} \mathbf{R}_n^b(\boldsymbol{\Theta}_{nb}) & \mathbf{0}_{3\times3} \\ \mathbf{0}_{3\times3} & \mathbf{T}_b^{-1}(\boldsymbol{\Theta}_{nb}) \end{bmatrix}$$

$\mathbf{J}_b^n(\boldsymbol{\eta})^{-1} \neq \mathbf{J}_b^n(\boldsymbol{\eta})^T$ Because $\mathbf{T}_b$ is not orthogonal.

Summary manoeuvring coordinates

Perez, T. and T.I. Fossen (2007)

Variable	Description
$\mathbf{r}^n_{nb} = [N, E, D]^T$	Vessel position in $\{n\}$
$\mathbf{v}^b_{nb} = [u, v, w]^T$	Vessel linear velocity in $\{b\}$
$\boldsymbol{\omega}^b_{nb} = [p, q, r]^T$	Vessel angular velocity in $\{b\}$
$\boldsymbol{\Theta}_{nb} = [\phi, \theta, \psi]^T$	Euler angles that take $\{n\}$ into $\{b\}$
$\boldsymbol{\eta} = [(\mathbf{r}^n_{nb})^T, (\boldsymbol{\Theta}_{nb})^T]^T$	Generalised position vector
$\boldsymbol{\nu} = [(\mathbf{v}^b_{nb})^T, (\boldsymbol{\omega}^b_{nb})^T]^T$	Generalised velocity vector
$\dot{\boldsymbol{\eta}} = \mathbf{J}^n_b(\boldsymbol{\eta})\boldsymbol{\nu}$	Vessel trajectory

Seakeeping kinematics

- In seakeeping, the motion is described from a reference frame which represents the equilibrium position and orientation of the vessel.
- Then the action of the waves makes the vessel oscillate with respect to this equilibrium

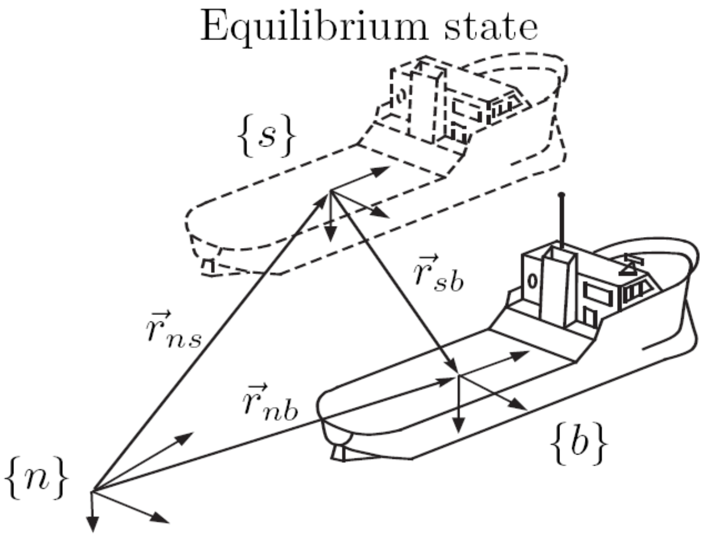
Definition of equilibrium reference frame:

$$\mathbf{v}_{ns}^{n} = \dot{\mathbf{r}}_{ns}^{n} = [U\cos\bar{\psi}, U\sin\bar{\psi}, 0]^T$$

$$\boldsymbol{\omega}_{ns}^{n} = [0,0,0]^T,$$

$$\boldsymbol{\Theta}_{ns} = [0,0,\bar{\psi}]^T,$$

$$\mathbf{v}_{ns}^{s} = \mathbf{R}_{n}^{s}\mathbf{v}_{ns}^{n} = [U,0,0]^T$$



$$U = \|\mathbf{v}_{ns}^{n}\| = \|{}^{n}d\vec{r}_{ns}/dt\|$$

Vessel average forward speed.

Seakeeping coordinates

In a similar fashion as we did in manoeuvring, we can define the perturbation body-fixed linear and angular velocities

$$
\begin{aligned}
\mathbf{v}_{sb}^{b} = \mathbf{R}_s^b \dot{\mathbf{r}}_{sb}^{s} &\triangleq [\delta u, \delta v, \delta w]^T, \\
\boldsymbol{\omega}_{sb}^{b} &\triangleq [\delta p, \delta q, \delta r]^T,
\end{aligned}
$$

Perturbation roll, pitch and yaw:

$$\boldsymbol{\Theta}_{sb} \triangleq [\delta\phi, \delta\theta, \delta\psi]^T$$

$$\dot{\boldsymbol{\Theta}}_{sb} = \mathbf{T}_b(\boldsymbol{\Theta}_{sb})\,\boldsymbol{\omega}_{sb}^{b},$$

$$\mathbf{T}_b(\boldsymbol{\Theta}_{sb}) = \begin{bmatrix} 1 & s_{\delta\phi} t_{\delta\theta} & c_{\delta\phi} t_{\delta\theta} \\ 0 & c_{\delta\phi} & -s\delta\phi \\ 0 & s_{\delta\phi}/c_{\delta\theta} & c_{\delta\phi}/c_{\delta\theta} \end{bmatrix}$$

Seakeeping coordinates

Further, we can define the perturbation position an velocity vector:

$$\delta\boldsymbol{\eta} \triangleq \begin{bmatrix} \mathbf{r}^s_{sb} \\ \boldsymbol{\Theta}_{sb} \end{bmatrix}, \qquad \delta\boldsymbol{\nu} \triangleq \begin{bmatrix} \mathbf{v}^b_{sb} \\ \boldsymbol{\omega}^b_{sb} \end{bmatrix}$$

In the hydrodynamic literature, the following variables are used:

$$\boldsymbol{\xi} \triangleq \delta\boldsymbol{\eta}$$

$$\dot{\boldsymbol{\xi}} = \delta\boldsymbol{\nu} \approx \delta\dot{\boldsymbol{\eta}}$$

The kinematics transformation is simplified under the assumptions of very small angles

Summary seakeeping coordinates

Perez, T. and T.I. Fossen (2007)

Variable	Description
$\mathbf{r}^s_{sb}$	Vessel perturbation displ. in $\{s\}$
$\mathbf{v}^b_{sb} = [\delta u, \delta v, \delta w]^T$	Vessel linear pert. velocity in $\{b\}$
$\boldsymbol{\omega}^b_{sb} = [\delta p, \delta q, \delta r]^T$	Vessel pert. angular vel. in $\{b\}$
$\boldsymbol{\Theta}_{sb} = [\delta\phi, \delta\theta, \delta\psi]^T$	Euler ang. that take $\{s\}$ into $\{b\}$
$\delta\boldsymbol{\eta} = [(\mathbf{r}^s_{sb})^T, (\boldsymbol{\Theta}_{sb})^T]^T$	Generalised pert. position vector
$\boldsymbol{\xi} = \delta\boldsymbol{\eta}$	Seakeeping variables
$\delta\boldsymbol{\nu} = [(\mathbf{v}^b_{sb})^T, (\boldsymbol{\omega}^b_{sb})^T]^T$	Generalised pert. velocity vector

Kinetics

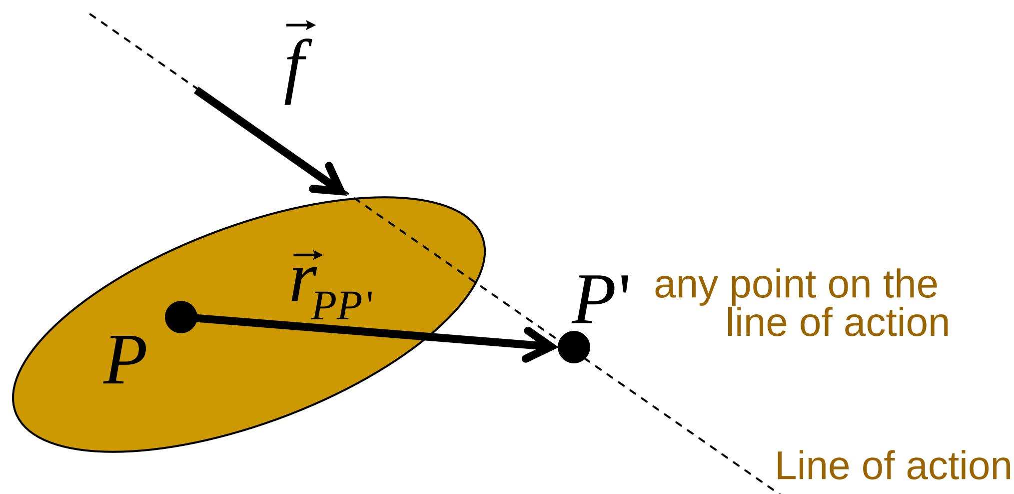
Description of forces and the motion they cause on bodies using postulated laws of physics.

Forces and moments

- A force acting on a rigid body has a line of action which passes through the point of application.

- This means that the force produces a moment about a point.

$$\vec{m}_{b/P} = \vec{r}_{PP'} \times \vec{f}$$



Forces and moments

- A resultant force due to a set S of forces acting on a rigid body is

$$\vec{f}_{RES} = \sum_j \vec{f}_j$$

The resultant does not have a line of action.

- A resultant moment due to a set S of forces acting on a rigid body is

$$\vec{m}_{S/P} = \sum_j \vec{r}_{Pj} \times \vec{f}_j$$

Moment about another point

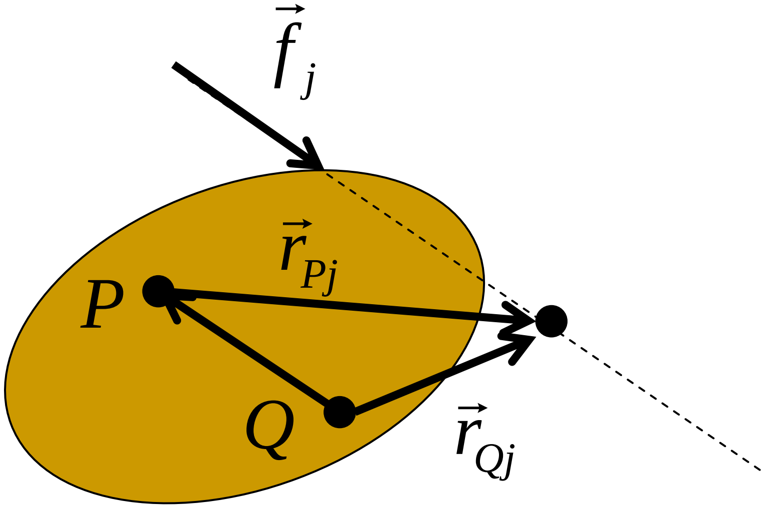
The moment about a point Q can be found

$$\vec{m}_{S/Q} = \sum_j \vec{r}_{Qj} \times \vec{f}_j = \sum_j (\vec{r}_{Pj} + \vec{r}_{QP}) \times \vec{f}_j$$

$$= \sum_j \vec{r}_{Pj} \times \vec{f}_j + \vec{r}_{QP} \times \sum_j \vec{f}_j$$

$$= \vec{m}_{S/P} + \underbrace{\vec{r}_{QP} \times \vec{f}_{RES}}$$

The resultant can be regarded as a force with line of action through P



Transformation

- Using the previous results we have that in body-fixed coordinates

$$\begin{bmatrix} \mathbf{f}_Q^b \\ \mathbf{m}_Q^b \end{bmatrix} = \begin{bmatrix} \mathbf{f}_P^b \\ \mathbf{S}(\mathbf{r}_{QP}^b)\,\mathbf{f}_P^b + \mathbf{m}_P^b \end{bmatrix} = \underbrace{\begin{bmatrix} \mathbf{I}_{3\times3} & \mathbf{0}_{3\times3} \\ \mathbf{S}(\mathbf{r}_{QP}^b) & \mathbf{I}_{3\times3} \end{bmatrix}}_{\mathbf{H}^T(\mathbf{r}_{QP}^b)} \begin{bmatrix} \mathbf{f}_P^b \\ \mathbf{m}_P^b \end{bmatrix}$$

If we choose Q = Ob, then

$$\begin{bmatrix} \mathbf{f}_b^b \\ \mathbf{m}_b^b \end{bmatrix} = \begin{bmatrix} \mathbf{f}_P^b \\ \mathbf{S}(\mathbf{r}_{bP}^b)\,\mathbf{f}_P^b + \mathbf{m}_P^b \end{bmatrix} = \underbrace{\begin{bmatrix} \mathbf{I}_{3\times3} & \mathbf{0}_{3\times3} \\ \mathbf{S}(\mathbf{r}_{bP}^b) & \mathbf{I}_{3\times3} \end{bmatrix}}_{\mathbf{H}^T(\mathbf{r}_{bP}^b)} \begin{bmatrix} \mathbf{f}_P^b \\ \mathbf{m}_P^b \end{bmatrix}$$

Rigid-body mass and inertia matrix

The generalised mass matrix with inertia moments and products taken about the origin of {b} is

$$\mathbf{M}_{RB}^{b} = \begin{bmatrix} m\mathbf{I}_{3\times 3} & -m\mathbf{S}(\mathbf{r}_{bg}^{b}) \\ m\mathbf{S}(\mathbf{r}_{bg}^{b}) & \mathbf{I}_{b/b}^{b} \end{bmatrix}$$

Inertia matrix taken about the origin {b} can be expressed by the one taken about CG (Parallel axis theorem):

$$\mathbf{I}_{b/b}^{b} = \mathbf{I}_{b/g}^{b} - m\mathbf{S}(\mathbf{r}_{bg}^{b})\mathbf{S}(\mathbf{r}_{bg}^{b})$$

$$\mathbf{I}_{b/g}^{b} = \int_{b} \begin{bmatrix} y^2+z^2 & -xy & -xz \\ -xy & x^2+z^2 & -yz \\ -xz & -yz & x^2+y^2 \end{bmatrix} dm$$

The notation "b/" means about; e.g. b/g means about CG

The parallel axis theorem can only be use between CG and another point. So if we want to convert between two arbitrary points we have to do it in two steps.

Angular momentum

The angular momentum about CG in B is given by

$$\mathbf{h}_g^b = \mathbf{I}_{b/g}^b \boldsymbol{\omega}_{ib}^b$$

If this is expressed in the inertial coordinate system {i},

$$\mathbf{h}_g^i = \mathbf{R}_b^i \mathbf{I}_{b/g}^b \mathbf{R}_i^b \boldsymbol{\omega}_{ib}^i \qquad \mathbf{I}_{b/g}^i = \mathbf{R}_b^i \mathbf{I}_{b/g}^b \mathbf{R}_i^b$$

This shows that inertia matrix is not constant in an inertial frame {i} if {b} rotates wrt {i}. Hence, it is convenient to express the equations of motion in a body-fixed coordinate system.

Euler’s Axioms

- Euler’s 1st axiom states that:

$$m\dot{\mathbf{v}}_{ig}^{i} = \mathbf{f}^{i}$$

$\mathbf{v}_{ig}^{i}$ -is the velocity of CG relative to {i} and expressed in {i}

$\mathbf{f}^{i}$ -is the vector of resultant forces.

- Euler’s 2nd axiom states that:

$$\dot{\mathbf{h}}_{g}^{i} = \mathbf{m}_{g}^{i}$$

Angular momentum about CG

Resultant moment about CG

Rigid-body Equations of motion

Expressing the velocities in the body-fixed coordinate system—located at an arbitrary point in the body, the Euler axioms become

$$m[\dot{\mathbf{v}}_{ib}^{b} + \mathbf{S}(\dot{\boldsymbol{\omega}}_{ib}^{b})\mathbf{r}_{bg}^{b} + \mathbf{S}(\boldsymbol{\omega}_{ib}^{b})\mathbf{v}_{ib}^{b} + \mathbf{S}^{2}(\boldsymbol{\omega}_{ib}^{b})\mathbf{r}_{bg}^{b}] = \mathbf{f}^{b}$$

$$\mathbf{I}_{b/b}^{b}\dot{\boldsymbol{\omega}}_{ib}^{b} + \mathbf{S}(\boldsymbol{\omega}_{ib}^{b})\mathbf{I}_{b/b}^{b}\boldsymbol{\omega}_{ib}^{b} + m\mathbf{S}(\mathbf{r}_{bg}^{b})\dot{\mathbf{v}}_{ib}^{b} + m\mathbf{S}(\mathbf{r}_{bg}^{b})\mathbf{S}(\boldsymbol{\omega}_{ib}^{b})\mathbf{v}_{ib}^{b} = \mathbf{m}_{b}^{b}$$

Ship RB eq of motion

Following the notation of Fossen (2002):

$$\mathbf{M}_{RB}^{b}\dot{\boldsymbol{\nu}} + \mathbf{C}_{RB}(\boldsymbol{\nu})\boldsymbol{\nu} = \boldsymbol{\tau}^{b}$$

$$\boldsymbol{\eta} \triangleq \begin{bmatrix} \mathbf{r}_{nb}^{n} \\ \boldsymbol{\Theta}_{nb} \end{bmatrix} = [N, E, D, \phi, \theta, \psi]^{T}$$

Generalised positions

$$\boldsymbol{\nu} \triangleq \begin{bmatrix} \mathbf{v}_{nb}^{b} \\ \boldsymbol{\omega}_{nb}^{b} \end{bmatrix} = [u, v, w, p, q, r]^{T}$$

Generalised velocities

$$\boldsymbol{\tau}^{b} := \begin{bmatrix} \mathbf{f}^{b} \\ \mathbf{m}_{b}^{b} \end{bmatrix} = [X, Y, Z, K, M, N]^{T}$$

Generalised forces

Coriolis and Centripetal terms

The Coriolis-centripetal terms can be expressed as

$$\mathbf{C}_{RB}(\boldsymbol{\nu}) \triangleq \begin{bmatrix} \mathbf{C}_{RB,11} & \mathbf{C}_{RB,12} \\ \mathbf{C}_{RB,21} & \mathbf{C}_{RB,22} \end{bmatrix}$$

where

$$\boldsymbol{\nu} = [\boldsymbol{\nu}_1^T, \boldsymbol{\nu}_2^T]^T$$

← Separated into linear and anglular

$$\begin{aligned}
\mathbf{C}_{RB,11} &= \mathbf{0}_{3\times 3}, \\
\mathbf{C}_{RB,12} &= -m\mathbf{S}(\boldsymbol{\nu}_1) - m\mathbf{S}(\mathbf{S}(\boldsymbol{\nu}_2)\mathbf{r}_{bg}^b), \\
\mathbf{C}_{RB,21} &= -m\mathbf{S}(\boldsymbol{\nu}_1) - m\mathbf{S}(\mathbf{S}(\boldsymbol{\nu}_2)\mathbf{r}_{bg}^b), \\
\mathbf{C}_{RB,22} &= m\mathbf{S}(\mathbf{S}(\boldsymbol{\nu}_1)\mathbf{r}_{bg}^b) - \mathbf{S}(\mathbf{I}_{b/b}^b\boldsymbol{\nu}_2),
\end{aligned}$$

RB Equations of motion in 6DOF

In coordinate form:

$$
\begin{aligned}
m\left[\dot u - vr + wq - x_g^b(q^2+r^2) + y_g^b(pq-\dot r) + z_g^b(pr+\dot q)\right] &= \tau_1^b \\
m\left[\dot v - wp + ur - y_g^b(r^2+p^2) + z_g^b(qr-\dot p) + x_g^b(qp+\dot r)\right] &= \tau_2^b \\
m\left[\dot w - uq + vp - z_g^b(p^2+q^2) + x_g^b(rp-\dot q) + y_g^b(rq+\dot p)\right] &= \tau_3^b \\
I_x^b\dot p + (I_z^b - I_y^b)qr - (\dot r + pq)I_{xz}^b + (r^2-q^2)I_{yz}^b + (pr-\dot q)I_{xy}^b & \\
+m\left[y_g^b(\dot w - uq + vp) - z_g^b(\dot v - wp + ur)\right] &= \tau_4^b \\
I_y^b\dot q + (I_x^b - I_z^b)rp - (\dot p + qr)I_{xy}^b + (p^2-r^2)I_{zx}^b + (qp-\dot r)I_{yz}^b & \\
+m\left[z_g^b(\dot u - vr + wq) - x_g^b(\dot w - uq + vp)\right] &= \tau_5^b \\
I_z^b\dot r + (I_y^b - I_x^b)pq - (\dot q + rp)I_{yz}^b + (q^2-p^2)I_{xy}^b + (rq-\dot p)I_{zx}^b & \\
+m\left[x_g^b(\dot v - wp + ur) - y_g^b(\dot u - vr + wq)\right] &= \tau_6^b
\end{aligned}
$$

Changing the body-fixed system

In different applications it is necessary to consider different locations for a body-fixed coordinate system.

Hence we may want to transform the equations of motion from {b} to {p}

$$\mathbf{M}_{RB}^{b}\dot{\boldsymbol{\nu}} + \mathbf{C}_{RB}(\boldsymbol{\nu})\boldsymbol{\nu} = \boldsymbol{\tau}^{b}$$

$$\Downarrow$$

$$\mathbf{M}_{RB}^{p}\dot{\boldsymbol{\nu}}^{p} + \mathbf{C}_{RB}^{p}(\boldsymbol{\nu}^{p})\boldsymbol{\nu}^{p} = \boldsymbol{\tau}^{p}$$

Changing the body-fixed coordinates

This can be done by direct application of the transformations we have already derived:

$$\begin{bmatrix} \mathbf{v}_{ip}^{p} \\ \boldsymbol{\omega}_{ip}^{p} \end{bmatrix} = \underbrace{\begin{bmatrix} \mathbf{I}_{3\times3} & \mathbf{S}^{T}(\mathbf{r}_{bp}^{b}) \\ \mathbf{0}_{3\times3} & \mathbf{I}_{3\times3} \end{bmatrix}}_{\mathbf{H}(\mathbf{r}_{bp}^{b})} \begin{bmatrix} \mathbf{v}_{ib}^{b} \\ \boldsymbol{\omega}_{ib}^{b} \end{bmatrix} \qquad \begin{bmatrix} \mathbf{f}_{b}^{b} \\ \mathbf{m}_{b}^{b} \end{bmatrix} = \underbrace{\begin{bmatrix} \mathbf{I}_{3\times3} & \mathbf{0}_{3\times3} \\ \mathbf{S}(\mathbf{r}_{bP}^{b}) & \mathbf{I}_{3\times3} \end{bmatrix}}_{\mathbf{H}^{T}(\mathbf{r}_{bP}^{b})} \begin{bmatrix} \mathbf{f}_{P}^{b} \\ \mathbf{m}_{P}^{b} \end{bmatrix}$$

Hence,

$$\begin{aligned} \boldsymbol{\nu}^{p} &= \mathbf{H}(\mathbf{r}_{bp}^{b})\boldsymbol{\nu} \\ \boldsymbol{\tau}^{p} &= \mathbf{H}^{-T}(\mathbf{r}_{bp}^{b})\boldsymbol{\tau} \\ \mathbf{M}_{RB}^{p} &= \mathbf{H}^{-T}(\mathbf{r}_{bp}^{b})\mathbf{M}_{RB}^{b}\mathbf{H}^{-1}(\mathbf{r}_{bp}^{b}) \\ \mathbf{C}_{RB}^{p}(\boldsymbol{\nu}^{p}) &= \mathbf{H}^{-T}(\mathbf{r}_{bp}^{b})\mathbf{C}_{RB}^{b}(\mathbf{H}^{-1}\boldsymbol{\nu}^{p})\mathbf{H}^{-1}(\mathbf{r}_{bp}^{b}) \end{aligned}$$

Seakeeping RB equations of motion

In Seakeeping, the equations of motion are considered within an linear framework.

In the literature, it is said that the motion is described from the equilibrium reference frame and formulated at the origin of {s}—seakeeping coordinate system.

This would imply that the inertia matrix is time varying as we have seen in the previous slide, but this is not the case.

Seakeeping RB equations of motion

The seakeeping RB eq. can be obtained by considering the equations of motion in body-fixed coordinates and considering only linear terms:

$$\delta\dot{\boldsymbol{\eta}} = \mathbf{J}_b^s(\delta\boldsymbol{\eta})\delta\boldsymbol{\nu},$$
$$\mathbf{M}_{RB}\delta\dot{\boldsymbol{\nu}} + \mathbf{C}_{RB}(\delta\boldsymbol{\nu})\delta\boldsymbol{\nu} = \delta\boldsymbol{\tau}$$

Perturbation Eq in body-fixed coord.

$$\Rightarrow$$

$$\delta\dot{\boldsymbol{\eta}} \approx \delta\boldsymbol{\nu}$$
$$\mathbf{M}_{RB}\delta\dot{\boldsymbol{\nu}} \approx \delta\boldsymbol{\tau}$$

Perturbation within Linear framework

Seakeeping RB Eq. of motion:

$$\mathbf{M}_{RB}\delta\ddot{\boldsymbol{\eta}} \approx \delta\boldsymbol{\tau}$$

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