Hydrodynamics for control engineers

(Module 2)

Dr Tristan Perez

Centre for Complex Dynamic Systems and Control (CDSC)

Prof. Thor I Fossen

Department of Engineering Cybernetics

Marine hydrodynamics

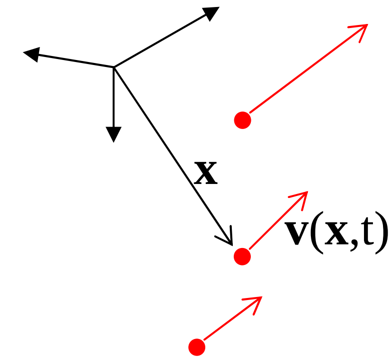
In order to study the motion of marine structures and vessels, we need to understand the effects the surrounding fluid has on them.

This requires some basic concepts of hydrodynamics—which is fluid dynamics under special-case simplifications and assumptions particular of marine applications.

To solve problems related to ship motion we, need to know two things about the fluid:

- velocity
- pressure

Fluid flow description



The velocity of the fluid at the location

$$\mathbf{x} = \left[x_1,\, x_2,\, x_3\right]^{\mathrm{T}}$$

is given by the fluid-flow velocity vector:

$$\mathbf{v}(\mathbf{x},t) = \left[v_1(\mathbf{x},t),\, v_2(\mathbf{x},t),\, v_3(\mathbf{x},t)\right]^{\mathrm{T}}$$

this vector is usually described relative to an inertial coordinate system with origin in the mean free surface (h-frame, or s-frame).

Incompressible fluid

For the flow velocities involved in ship motion, the fluid can be considered *incompressible*, *i.e.,* constant density.

Under this assumption, the net volume rate at a volume V enclosed by a surface S is

$$\iint_S \mathbf{v} \cdot \mathbf{n}\, ds = \iiint_V \mathbf{div}(\mathbf{v})\, dV = 0$$

since this is valid for all the regions V in the fluid, then by assuming that $\mathbf{div}(\mathbf{v})$ is continuous, we obtain the *continuity equation* for incompressible flows:

$$\mathbf{div}(\mathbf{v}) = \nabla \cdot \mathbf{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z} = 0$$

Material derivative

Let $f(x,y,z,t)$ be a scalar function and $\mathbf{f}(t,x,y,z)$ a vector-valued function; then,

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$$

$$\frac{d\mathbf{f}}{dt} = \frac{\partial \mathbf{f}}{\partial t} + \frac{\partial \mathbf{f}}{\partial x}\frac{dx}{dt} + \frac{\partial \mathbf{f}}{\partial y}\frac{dy}{dt} + \frac{\partial \mathbf{f}}{\partial z}\frac{dz}{dt}.$$

If these are taken for the function $\mathbf{x}(t)$ s.t. $\dot{\mathbf{x}}(t) = \mathbf{v}(t)$

then we have a special notation—material derivative:

$$\frac{Df}{Dt} = \frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f, \qquad \frac{D\mathbf{f}}{Dt} = \frac{\partial \mathbf{f}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{F}$$

Flow equations

The conservation of momentum in the flow is described by the

Navier-Stokes (N-S) Equation:

$$\rho \frac{D\mathbf{v}}{Dt} = \rho\,\mathbf{F} - \nabla p + \mu \nabla^2 \mathbf{v}$$

$\mathbf{F}$ are accelerations due to volumetric forces: $\mathbf{F} = [0\,0\,-g]^{\mathrm{T}}$

$p = p(\mathbf{x}, t)$ is the pressure, and μ is the viscosity of the fluid.

- Unknowns: $\mathbf{v}$ and p
- N-S + Continuity eq. form a system of Nonlinear PDE
- No analytical solution exists for realistic ship flows.
- Numerical solutions are still far from feasible
- Practical approaches: RANS (CFD)

Potential theory

A further simplification is obtained by assuming that the fluid is *inviscid* and the flow is *irrotational.* Irrotaional means that

$$\mathbf{curl}(\mathbf{v}) = \nabla \times \mathbf{v} = 0$$

Under this assumption, then exists a scalar function Φ called *potential* such that

$$\mathbf{v} = \nabla \Phi$$

So, if we know the potential, we can calculate the flow velocity vector (the gradient of the potential).

How do we obtain the potential?

In potential theory, the continuity equation reverts to the *Laplacian* of the potential equal to zero:

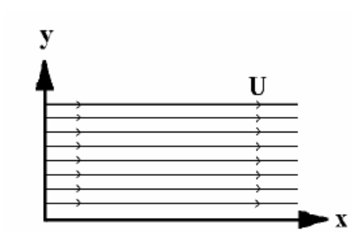
$$\nabla^2 \Phi = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0$$

The potential is, thus, obtained by solving this subject to appropriate boundary conditions, *i.e.*, by solving a boundary value problem (VBP).

The Laplace Equation is linear $\Leftrightarrow$ Superposition of flows.

Ex:Potential Flow Superposition

Uniform



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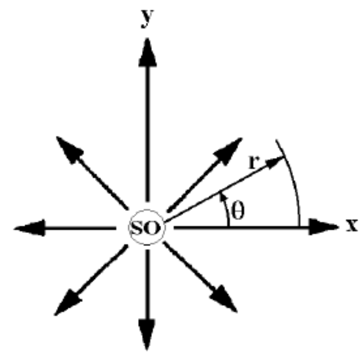
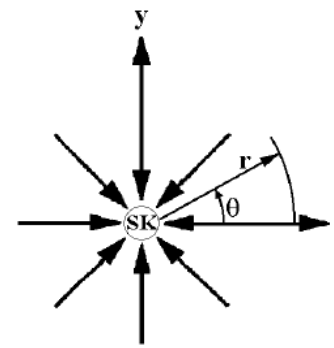
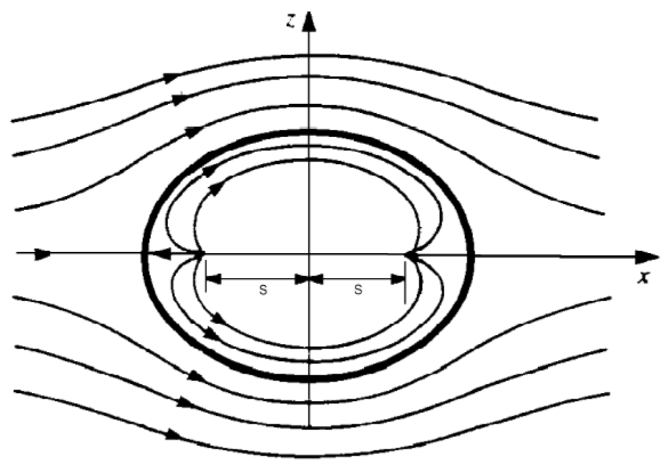


Sink





How do we calculate pressure?

If we neglect viscosity in the N-S equation, we obtain the Euler Equation of flow:

$$\rho \frac{D\mathbf{v}}{Dt} = \rho\,\mathbf{F} - \nabla p$$

Then,

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\nabla \left(\frac{p}{\rho} + \Upsilon \right)$$

where $$-\nabla \Upsilon = \mathbf{F},\ i.e.\ \Upsilon = gz$$

Irrotaional flow assumption

Using some vector calculus

$$\frac{\partial \mathbf{v}}{\partial t} + (\nabla \times \mathbf{v}) \times \mathbf{v} = -\nabla \left(\frac{p}{\rho} + \frac{1}{2}\mathbf{v}^2 + \Upsilon \right)$$

If the flow is irrotational, then

$$\nabla \left(\frac{p}{\rho} + \frac{\partial \Phi}{\partial t} + \frac{1}{2}(\nabla \Phi)^2 + \Upsilon \right) = 0$$

Bernoulli equation

$$\nabla\left(\frac{p}{\rho}+\frac{\partial\Phi}{\partial t}+\frac{1}{2}(\nabla\Phi)^2+\Upsilon\right)=0$$

If this is valid in the whole fluid, then

$$\frac{p}{\rho}+\frac{\partial\Phi}{\partial t}+\frac{1}{2}(\nabla\Phi)^2+gz=C$$

which is the Bernoulli equation.

Potential theory—summary

Potential theory offers a great simplification: if we know the potential, then we know the velocity and the pressure, from which we can calculate the forces acting on a floating body by integrating the pressure over the surface of the body.

Inviscid fluid and *irrotational flow*

↓

Potential

↙ ↘

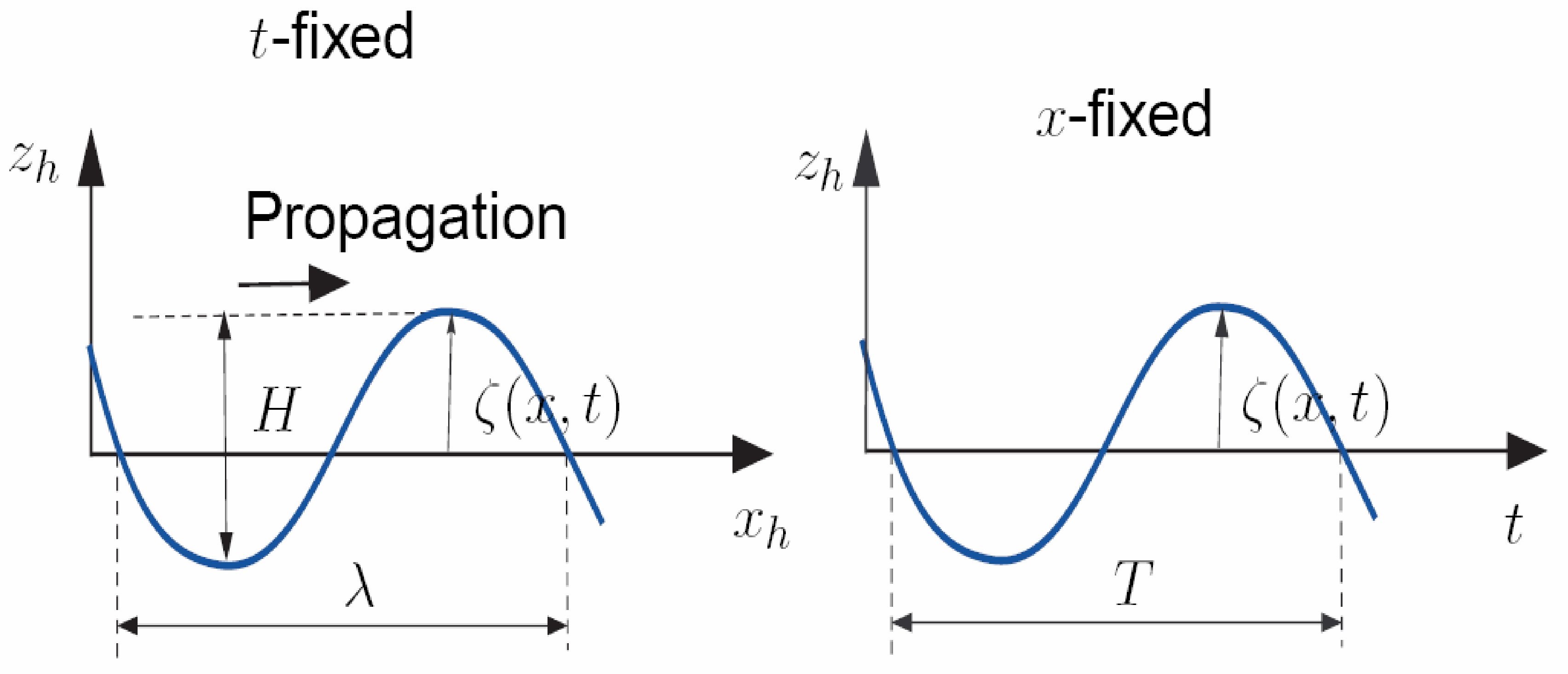
Flow velocity *Pressure*

for *most* problems related ship motion in waves, potential theory is sufficient for engineering purposes. Viscous effects are added to the models using empirical formulae or via system identification.

Applications

- Regular waves
- Marine structures in waves

Regular waves in deep water



The sea surface elevation is denoted by $\zeta(x, t)$.

Kinematic free-surface Condition

Kinematic free-surface condition: A fluid particle on the free surface is assumed to remain on the free surface.

Let the free surface be defined as $z = \zeta(x, y, t)$

Then, if $F := z - \zeta(x, y, t)$

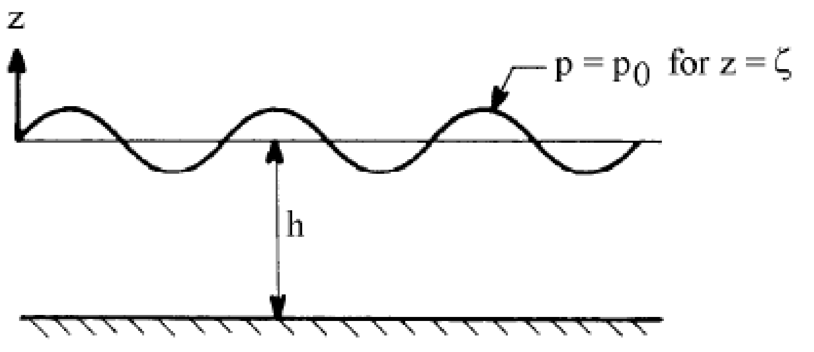
The kinematic condition reverts to $\dfrac{DF}{Dt} = 0$

Hence,

$$\frac{\partial \zeta}{\partial t} + \frac{\partial \phi}{\partial x}\frac{\partial \zeta}{\partial x} + \frac{\partial \phi}{\partial y}\frac{\partial \zeta}{\partial y} - \frac{\partial \phi}{\partial z} = 0 \qquad \text{on} \qquad z = \zeta(x, y, t)$$

Dynamic free-surface Conditions

Dynamic free-surface condition: the water pressure equals the atmospheric pressure on the free surface.



If we choose the constant in the Bernoulli equation as $C = p_0 / \rho$

Then,

$$g\zeta + \frac{\partial \phi}{\partial t} + \frac{1}{2}\left[\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2 + \left(\frac{\partial \phi}{\partial z}\right)^2\right] = 0 \qquad \text{on} \qquad z = \zeta(x, y, t)$$

Linearised free-surface conditions

The free-surface conditions can be linearised about the mean free-surface:

$$\frac{\partial \zeta}{\partial t} - \frac{\partial \phi}{\partial z} = 0$$

$$g\zeta + \frac{\partial \phi}{\partial t} = 0 \qquad \text{on} \qquad z = 0$$

Combined:

$$g\frac{\partial \phi}{\partial z} + \frac{\partial^2 \phi}{\partial t^2} = 0 \qquad \text{on} \qquad z = 0$$

Regular Wave linear BVP

Hence, the boundary value problem (BVP) is to find the potential, $\Phi_w(\mathbf{x},t)$ that satisfies

$$\nabla^2\Phi_w = 0$$

$$s.t. \quad \frac{\partial\Phi_w}{\partial z} = \frac{\partial\zeta}{\partial t} \quad \text{on} \quad z=0,$$

$$\frac{\partial\Phi_w}{\partial t} = -g\zeta \quad \text{on} \quad z=0,$$

$$\frac{\partial\Phi_w}{\partial z} = 0 \quad \text{on} \quad z=-h,$$

$$\zeta = \bar{\zeta}\sin(\omega t - kx + \varepsilon)$$

The linear free-surface $\Rightarrow$ solution will be valid for waves with small stepness, *i.e.*, $\bar{\zeta}/\lambda \ll 1$.

Regular Wave Potential

Solution:

- Deep water:

$$\Phi_w = \frac{g\bar{\zeta}}{\omega} e^{kz} \cos(\omega t - kx + \varepsilon)$$

- Shallow water:

$$\Phi_w = \frac{g\bar{\zeta}}{\omega} \frac{\cosh[k(z+h)]}{\cosh[kh]} \cos(\omega t - kx + \varepsilon)$$

We then have the velocities and pressure in the whole fluid domain:

$$[u, v, w]^{\mathsf{T}} = \nabla \Phi_w, \qquad p - p_0 = -\rho g z - \rho \frac{\partial \Phi}{\partial t}.$$

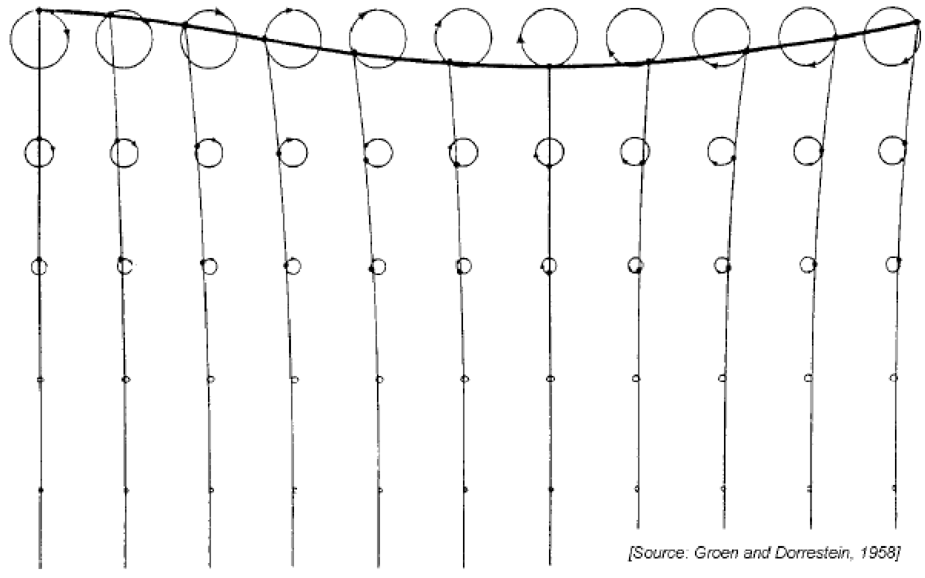
Regular wave formulae (Faltinsen, 1990)

	Finite water depth	Infinite water depth
Velocity potential	$\phi = \frac{g\zeta_a}{\omega}\frac{\cosh k(z+h)}{\cosh kh}\cos(\omega t - kx)$	$\phi = \frac{g\zeta_a}{\omega} e^{kz}\cos(\omega t - kx)$
Connection between wave number k and circular frequency ω	$\frac{\omega^2}{g} = k\tanh kh$	$\frac{\omega^2}{g} = k$ $\quad c = \frac{\omega}{k}$
Connection between wavelength λ and wave period T	$\lambda = \frac{g}{2\pi}T^2\tanh\frac{2\pi}{\lambda}h$	$\lambda = \frac{g}{2\pi}T^2$
Wave profile	$\zeta = \zeta_a\sin(\omega t - kx)$	$\zeta = \zeta_a\sin(\omega t - kx)$
Dynamic pressure	$p_D = \rho g\zeta_a\frac{\cosh k(z+h)}{\cosh kh}\sin(\omega t - kx)$	$p_D = \rho g\zeta_a e^{kz}\sin(\omega t - kx)$
x-component of velocity	$u = \omega\zeta_a\frac{\cosh k(z+h)}{\sinh kh}\sin(\omega t - kx)$	$u = \omega\zeta_a e^{kz}\sin(\omega t - kx)$
z-component of velocity	$w = \omega\zeta_a\frac{\sinh k(z+h)}{\sinh kh}\cos(\omega t - kx)$	$w = \omega\zeta_a e^{kz}\cos(\omega t - kx)$
x-component of acceleration	$a_1 = \omega^2\zeta_a\frac{\cosh k(z+h)}{\sinh kh}\cos(\omega t - kx)$	$a_1 = \omega^2\zeta_a e^{kz}\cos(\omega t - kx)$
z-component of acceleration	$a_3 = -\omega^2\zeta_a\frac{\sinh k(z+h)}{\sinh kh}\sin(\omega t - kx)$	$a_3 = -\omega^2\zeta_a e^{kz}\sin(\omega t - kx)$

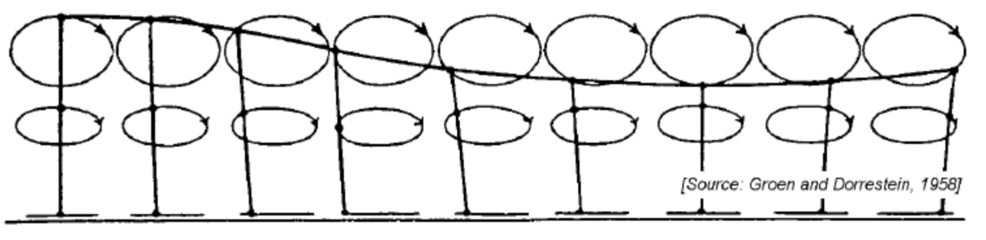
$\omega = 2\pi/T$, $k = 2\pi/\lambda$, $T =$ Wave period, $\lambda =$ Wavelength, $\zeta_a =$ Wave amplitude, $g =$ Acceleration of gravity, $t =$ Time variable, $x =$ direction of wave propagation, $z =$ vertical coordinate, z positive upwards, $z = 0$ mean waterlevel, $h =$ average waterdepth. Total pressure in the fluid: $p_D - \rho g z + p_0$ ($p_0 =$ atmospheric pressure).

Water particle trajectories

Deep water:



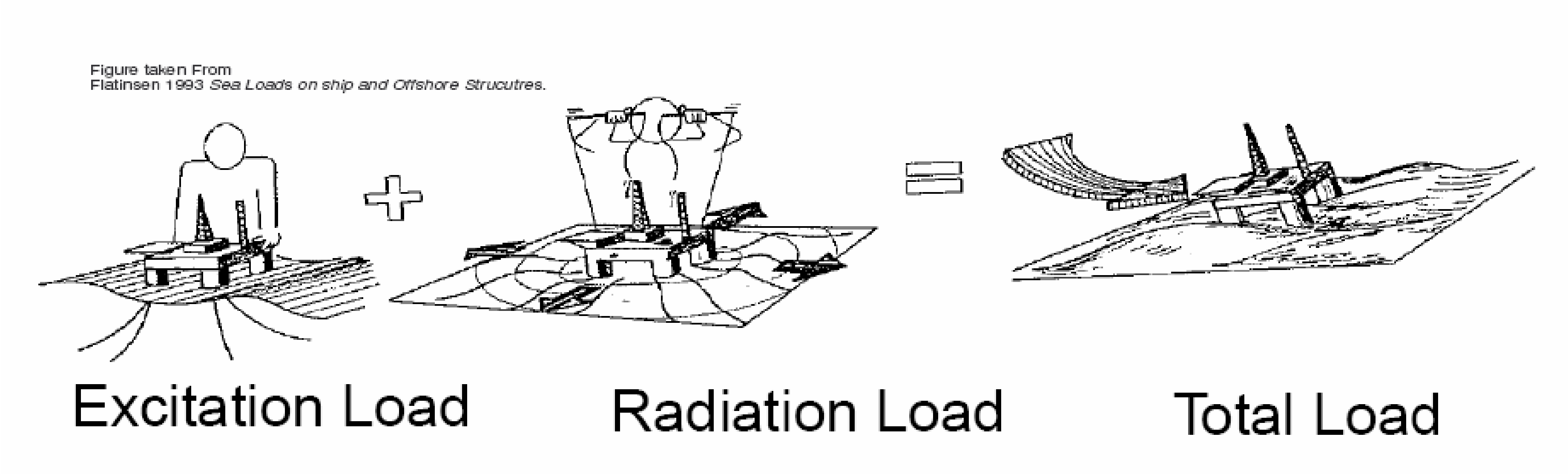
Shallow water:



Potential theory for ships in waves

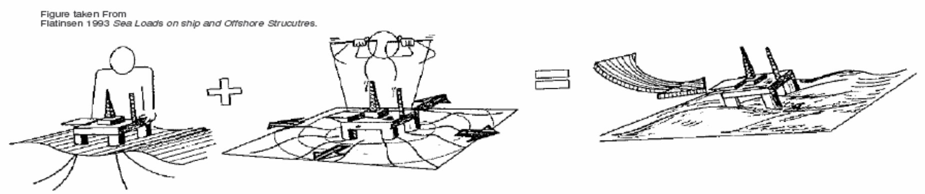
The fluid forces are due to variations in pressure on the surface of the hull.

It is normally assumed that the forces (pressure) can be made of different components



Potential theory for ships in waves

Under linearity assumptions, the hydrodynamic problem is dealt as 2 separate problems and the solutions then added:



- Radiation problem: the ship is forced to oscillate in calm water.

- Diffraction problem: the ship is restrained from moving in the presence of a wave field.

Potentials:

$$\Phi_{Total} = \underbrace{\sum_{j=1}^{6} \Phi_j}_{Radiation\ problem} + \underbrace{\Phi_{Incident} + \Phi_{Scattering}}_{Diffraction\ problem}$$

Φ_j - due to the motion in the j-th DOF.

Radiation potential

Boundary conditions:

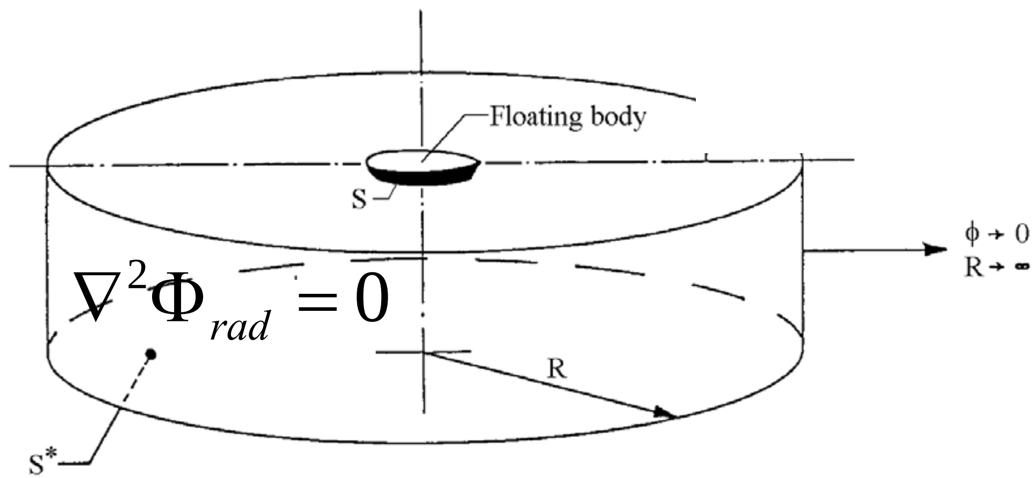
$$\frac{\partial^2 \Phi}{\partial t^2} + g \cdot \frac{\partial \Phi}{\partial z} = 0 \qquad \text{for: } z = 0$$
free surface condition (dynamic+kinematic conditions)

$$\frac{\partial \Phi}{\partial z} = 0 \qquad \text{for: } z = -h$$
sea bed condition

$$\frac{\partial \Phi}{\partial n} = v_n(x, y, z, t)$$
dynamic body condition

$$\lim_{R \to \infty} \Phi = 0$$
radiation condition

$$\Phi_{rad} = \sum_{j=1}^{6} \Phi_j$$



$$\nabla^2 \Phi_{rad} = 0$$

Computing forces

Forces and moments are obtained by integrating the pressure over the average wetted surface *Sw*:

Notation: i-th component

Radiation forces and moments:

DOF:

$$\tau^h_{\mathrm{r}i} = \begin{cases} -\iint_{Sw}\left(\frac{\partial\Phi_{\mathrm{r}}}{\partial t}\right)(\mathbf{n})_i\,ds & i = 1,2,3. \\ -\iint_{Sw}\left(\frac{\partial\Phi_{\mathrm{r}}}{\partial t}\right)(\mathbf{r}\times\mathbf{n})_{i-3}\,ds & i = 4,5,6. \end{cases}$$

1-surge
2-sway
3-heave
4-roll
5-pitch
6-yaw

Excitation forces (due to incident and scattered potentials) and moments:

$$\tau^h_{1\mathrm{w}i} = \begin{cases} -\iint_{Sw}\left(\frac{\partial\Phi_{1\mathrm{w}}}{\partial t}\right)(\mathbf{n})_i\,ds & i = 1,2,3. \\ -\iint_{Sw}\left(\frac{\partial\Phi_{1\mathrm{w}}}{\partial t}\right)(\mathbf{r}\times\mathbf{n})_{i-3}\,ds & i = 4,5,6. \end{cases}$$

References

- Faltinsen, O.M. (1990) **Sea Loads on Ships and Ocean Structures**. Cambridge University Press.
- Journée, J.M.J. and W.W. Massie (2001) **Offshore Hydromechanics.** Lecture notes on offshore hydromechanics for Offshore Technology students, code OT4620. (http://www.ocp.tudelft.nl/mt/journee/)