ETH zürich

# Lecture «Robot Dynamics»: Intro to Dynamics

**151-0851-00 V**

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|---|---|---|
| lecture: | CAB G11 | Tuesday 10:15 – 12:00, every week |
| exercise: | HG E1.2 | Wednesday 8:15 – 10:00, according to schedule (about every 2nd week) |

Marco Hutter, Roland Siegwart, and Thomas Stastny

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|---|---|---|---|---|---|
| 19.09.2017 | Intro and Outline | Course Introduction; Recapitulation Position, Linear Velocity | | | |
| 26.09.2017 | Kinematics 1 | Rotation and Angular Velocity; Rigid Body Formulation, Transformation | 26.09.2017 | Exercise 1a | Kinematics Modeling the ABB arm |
| 03.10.2017 | Kinematics 2 | Kinematics of Systems of Bodies; Jacobians | 03.10.2017 | Exercise 1b | Differential Kinematics of the ABB arm |
| 10.10.2017 | Kinematics 3 | Kinematic Control Methods: Inverse Differential Kinematics, Inverse Kinematics; Rotation Error; Multi-task Control | 10.10.2017 | Exercise 1c | Kinematic Control of the ABB Arm |
| 17.10.2017 | Dynamics L1 | Multi-body Dynamics | 17.10.2017 | Exercise 2a | Dynamic Modeling of the ABB Arm |
| 24.10.2017 | Dynamics L2 | Floating Base Dynamics | 24.10.2017 | | |
| 31.10.2017 | Dynamics L3 | Dynamic Model Based Control Methods | 31.10.2017 | Exercise 2b | Dynamic Control Methods Applied to the ABB arm |
| 07.11.2017 | Legged Robot | Dynamic Modeling of Legged Robots & Control | 07.11.2017 | Exercise 3 | Legged robot |
| 14.11.2017 | Case Studies 1 | Legged Robotics Case Study | 14.11.2017 | | |
| 21.11.2017 | Rotorcraft | Dynamic Modeling of Rotorcraft & Control | 21.11.2017 | Exercise 4 | Modeling and Control of Multicopter |
| 28.11.2017 | Case Studies 2 | Rotor Craft Case Study | 28.11.2017 | | |
| 05.12.2017 | Fixed-wing | Dynamic Modeling of Fixed-wing & Control | 05.12.2017 | Exercise 5 | Fixed-wing Control and Simulation |
| 12.12.2017 | Case Studies 3 | Fixed-wing Case Study (Solar-powered UAVs - AtlantikSolar, Vertical Take-off and Landing UAVs – Wingtra) | | | |
| 19.12.2017 | Summery and Outlook | Summery; Wrap-up; Exam | | | |

# Recapitulation of Kinematics

- Kinematics = **description** of motions
  - Translations and rotations
  - Various representations (Euler, quaternions, etc.)
  - Instantaneous/Differential kinematics
  - Jacobians and geometric Jacobians
  - Inverse kinematics and control
  - Floating base systems (unactuated base and contacts)

$$\boldsymbol{\chi}_e = \boldsymbol{\chi}_e(\mathbf{q})$$

$$\mathbf{w}_e = \begin{pmatrix} \mathbf{v}_e \\ \boldsymbol{\omega}_e \end{pmatrix} = \mathbf{J}_{e0}(\mathbf{q})\,\dot{\mathbf{q}}.$$

# Dynamics in Robotics

# Dynamics in Robotics

# Dynamics
## Outline

$$\mathbf{M}(\mathbf{q})\ddot{\mathbf{q}}+\mathbf{b}(\mathbf{q},\dot{\mathbf{q}})+\mathbf{g}(\mathbf{q})=\boldsymbol{\tau}+\mathbf{J}_c^T\mathbf{F}_c$$

| | |
|---|---|
| $\ddot{\mathbf{q}}$ | Generalized coordinates |
| $\mathbf{M}(\mathbf{q})$ | Mass matrix |
| $\mathbf{b}(\mathbf{q},\dot{\mathbf{q}})$ | Centrifugal and Coriolis forces |
| $\mathbf{g}(\mathbf{q})$ | Gravity forces |
| $\boldsymbol{\tau}$ | Generalized forces |
| $\mathbf{F}_c$ | External forces |
| $\mathbf{J}_c$ | Contact Jacobian |

- Description of “cause of motion”
  - Input $\boldsymbol{\tau}$ Force/Torque acting on system
  - Output $\ddot{\mathbf{q}}$ Motion of the system
- Principle of virtual work
  - Newton’s law for particles
  - ➢ Conservation of impulse and angular momentum
- 3 methods to get the EoM
  - Newton-Euler: Free cut and conservation of impulse & angular momentum for each body
  - Projected Newton-Euler (generalized coordinates)
  - Lagrange II (energy)
- Introduction to dynamics of floating base systems
  - External forces

# Principle of Virtual Work

- Principle of virtual work (D'Alembert's Principle)
  - Dynamic equilibrium imposes zero virtual work

$$\delta W = \int_{\mathcal{B}} \delta \mathbf{r}^T \cdot (\ddot{\mathbf{r}} \mathrm{d}m - \mathrm{d}\mathbf{F}_{ext}) = 0$$

*variational parameter*

| | |
|---|---|
| $\mathrm{d}\mathbf{F}_{ext}$ | external forces acting on element $i$ |
| $\ddot{\mathbf{r}}$ | acceleration of element $i$ |
| $dm$ | mass of element $i$ |
| $\delta \mathbf{r}$ | virtual displacement of element $i$ |

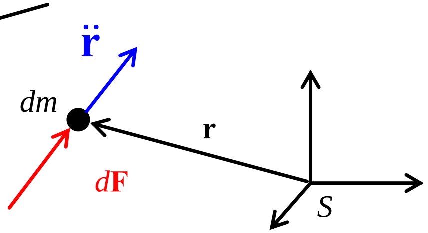
  - Newton's law for every particle in direction it can move

$$\mathbf{F} = m\mathbf{a} \qquad \mathbf{r} \times \mathbf{F} = \mathbf{r} \times m\mathbf{a}$$

$$\mathbf{F} = \frac{d}{dt}\Big(\underbrace{m\mathbf{v}}_{\mathbf{p}}\Big) = \dot{\mathbf{p}}$$

force

Impulse or linear momentum




$$\mathbf{\Gamma} = \frac{d}{dt}\Big(\underbrace{\mathbf{r} \times m\mathbf{v}}_{\mathbf{N}}\Big) = \mathbf{N}$$

moment

angular momentum

# Virtual Displacements of Single Rigid Bodies

- Rigid body Kinematics

$$\mathbf{r} = \mathbf{r}_{OS} + \boldsymbol{\rho}$$

$$\dot{\mathbf{r}} = \mathbf{v}_S + \boldsymbol{\Omega} \times \boldsymbol{\rho} = \begin{bmatrix} \mathbb{I}_{3\times 3} & -[\boldsymbol{\rho}]_\times \end{bmatrix} \begin{pmatrix} \mathbf{v}_s \\ \boldsymbol{\Omega} \end{pmatrix}$$

$$\ddot{\mathbf{r}} = \mathbf{a}_S + \Psi \times \boldsymbol{\rho} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{\rho}) = \begin{bmatrix} \mathbb{I}_{3\times 3} & -[\boldsymbol{\rho}]_\times \end{bmatrix} \begin{pmatrix} \mathbf{a}_s \\ \boldsymbol{\Psi} \end{pmatrix} + [\boldsymbol{\Omega}]_\times [\boldsymbol{\Omega}]_\times \boldsymbol{\rho}$$

$$\delta\mathbf{r} = \delta\mathbf{r}_S + \boldsymbol{\delta\Phi} \times \boldsymbol{\rho} = \begin{bmatrix} \mathbb{I}_{3\times 3} & -[\boldsymbol{\rho}]_\times \end{bmatrix} \begin{pmatrix} \delta\mathbf{r}_s \\ \boldsymbol{\delta\Phi} \end{pmatrix}$$

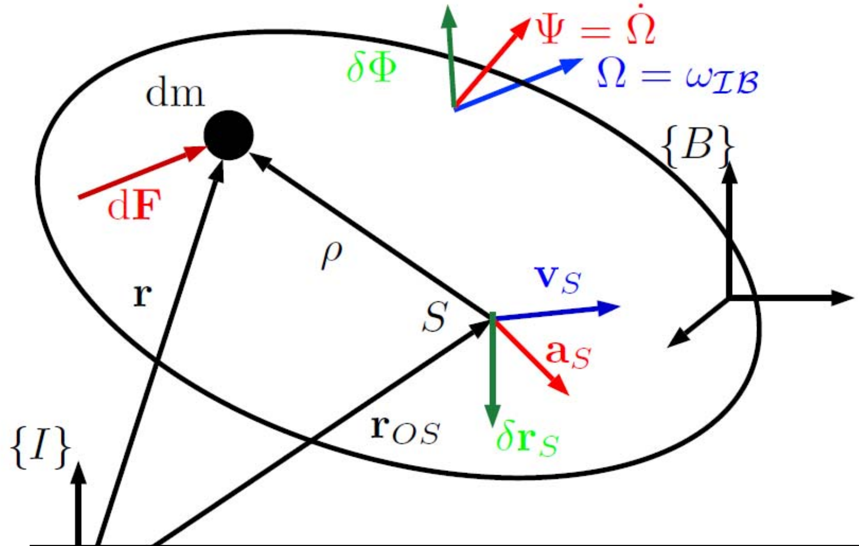
- Applied to principle of virtual work

$$0 = \delta W = \int_{\mathcal{B}} \begin{pmatrix} \delta\mathbf{r}_s \\ \delta\boldsymbol{\Phi} \end{pmatrix}^T \begin{bmatrix} \mathbb{I}_{3\times 3} \\ [\boldsymbol{\rho}]_\times \end{bmatrix} \left( \begin{bmatrix} \mathbb{I}_{3\times 3} & -[\boldsymbol{\rho}]_\times \end{bmatrix} \begin{pmatrix} \mathbf{a}_s \\ \boldsymbol{\Psi} \end{pmatrix} \mathrm{d}m + [\boldsymbol{\Omega}]_\times^2 \boldsymbol{\rho} \mathrm{d}m - \mathrm{d}\mathbf{F}_{ext} \right)$$

$$= \begin{pmatrix} \delta\mathbf{r}_s \\ \delta\boldsymbol{\Phi} \end{pmatrix}^T \int_{\mathcal{B}} \left( \begin{bmatrix} \mathbb{I}_{3\times 3}\mathrm{d}m & [\boldsymbol{\rho}]_\times^T \mathrm{d}m \\ [\boldsymbol{\rho}]_\times \mathrm{d}m & -[\boldsymbol{\rho}]_\times^2 \mathrm{d}m \end{bmatrix} \begin{pmatrix} \mathbf{a}_s \\ \boldsymbol{\Psi} \end{pmatrix} + \begin{pmatrix} [\boldsymbol{\Omega}]_\times^2 \boldsymbol{\rho}\mathrm{d}m \\ [\boldsymbol{\rho}]_\times [\boldsymbol{\Omega}]_\times^2 \boldsymbol{\rho}\mathrm{d}m \end{pmatrix} - \begin{pmatrix} \mathrm{d}\mathbf{F}_{ext} \\ [\boldsymbol{\rho}]_\times \mathrm{d}\mathbf{F}_{ext} \end{pmatrix} \right)$$

$$\boxed{0 = \delta W = \begin{pmatrix} \delta\mathbf{r}_s \\ \delta\boldsymbol{\Phi} \end{pmatrix}^T \left( \begin{bmatrix} \mathbb{I}_{3\times 3} m & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Theta}_S \end{bmatrix} \begin{pmatrix} \mathbf{a}_s \\ \boldsymbol{\Psi} \end{pmatrix} + \begin{pmatrix} \mathbf{0} \\ [\boldsymbol{\Omega}]_\times \boldsymbol{\Theta}_S \boldsymbol{\Omega} \end{pmatrix} - \begin{pmatrix} \mathbf{F}_{ext} \\ \mathbf{T}_{ext} \end{pmatrix} \right) \quad \forall \begin{pmatrix} \delta\mathbf{r}_s \\ \delta\boldsymbol{\Phi} \end{pmatrix}}$$

$$\delta W = \int_{\mathcal{B}} \delta\mathbf{r}^T \cdot (\ddot{\mathbf{r}}\mathrm{d}m - \mathrm{d}\mathbf{F}_{ext}) = 0$$




| | |
|---|---|
| $\int_{\mathcal{B}} \mathrm{d}m =: m$ | body mass |
| $\int_{\mathcal{B}} \boldsymbol{\rho}\mathrm{d}m =: \mathbf{0}$ | since S = COG |
| $\int_{\mathcal{B}} [\boldsymbol{\rho}]_\times^2 \mathrm{d}m = \int_{\mathcal{B}} [\boldsymbol{\rho}]_\times [\boldsymbol{\rho}]_\times \mathrm{d}m =: \boldsymbol{\Theta}_S$ | Inertia matrix around COG |

# Impulse and angular momentum

- Use the following definitions

| | |
|---|---|
| $\mathbf{p}_S = m\mathbf{v}_S$ | linear momentum |
| $\mathbf{N}_S = \mathbf{\Theta}_S\mathbf{\Omega}_S$ | angular momentum |
| $\dot{\mathbf{p}}_S = m\mathbf{a}_S$ | change in linear momentum |
| $\dot{\mathbf{N}}_S = \mathbf{\Theta}_S\mathbf{\Psi} + \mathbf{\Omega} \times \mathbf{\Theta}_S\mathbf{\Omega}$ | change in angular momentum |

- Conservation of impulse and angular momentum

Newton

$$0 = \delta W = \begin{pmatrix} \delta\mathbf{r}_s \\ \delta\mathbf{\Phi} \end{pmatrix}^T \left( \begin{pmatrix} \dot{\mathbf{p}}_S \\ \dot{\mathbf{N}}_S \end{pmatrix} - \begin{pmatrix} \mathbf{F}_{ext} \\ \mathbf{T}_{ext} \end{pmatrix} \right) \quad \forall \begin{pmatrix} \delta\mathbf{r}_s \\ \delta\mathbf{\Phi} \end{pmatrix}$$

A free body can move
In all directions

$$\boxed{\begin{aligned} \dot{\mathbf{p}}_S &= \mathbf{F}_{ext} \\ \dot{\mathbf{N}}_S &= \mathbf{T}_{ext} \end{aligned}}$$

Euler

External forces and moments

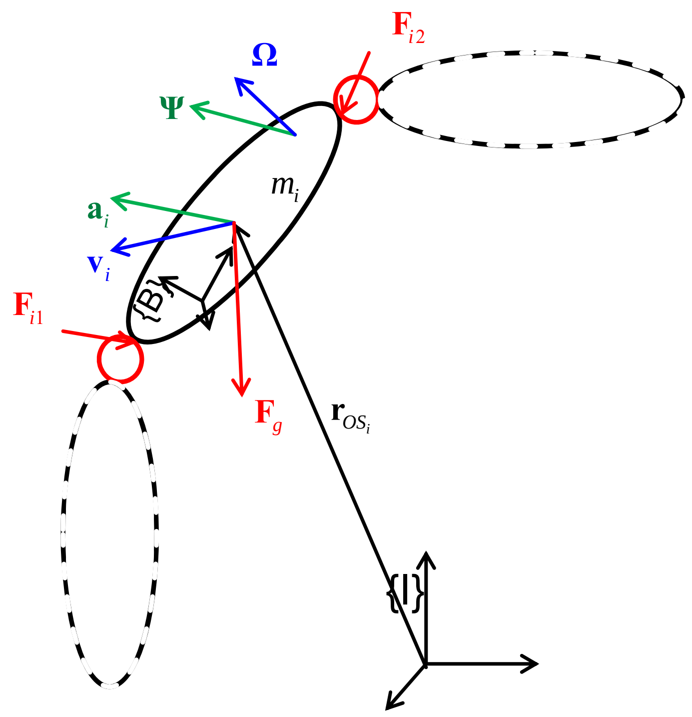
Change in impulse and angular momentum

# $1^{st}$ Method for EoM
## Newton-Euler for single bodies

- Cut all bodies free
- Introduction of constraining force
- Apply conservation $\mathbf{p}$ and $\mathbf{N}$ to individual bodies

➢ System of equations
  - 6n equation
  - Eliminate all constrained forces (5n)

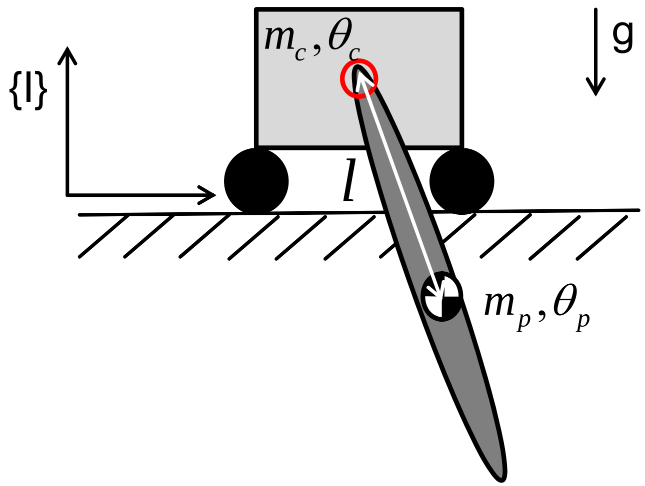
- Pros and Cons
  - \+ Intuitively clear
  - \+ Direct access to constraining forces
  - − Becomes a huge combinatorial problem for large MBS



# Free Cut

## Cart pendulum example

- Find the equation of motion



# Free Cut
## Cart pendulum

- Impulse / angular momentum cart

$$m_c \ddot{x}_c = F_x \quad (1)$$
$$m_c \ddot{y}_c = F_y + F_l + F_r - m_c g \quad (2)$$
$$\theta_c \ddot{\varphi}_c = F_r b - F_l b \quad (3)$$

- Impulse / angular momentum pendulum

$$m_p \ddot{x}_p = -F_x \quad (4)$$
$$m_p \ddot{y}_p = -F_y - m_p g \quad (5)$$
$$\theta_p \ddot{\varphi}_p = F_x l \cos(\varphi_p) + F_y l \sin(\varphi_p) \quad (6)$$

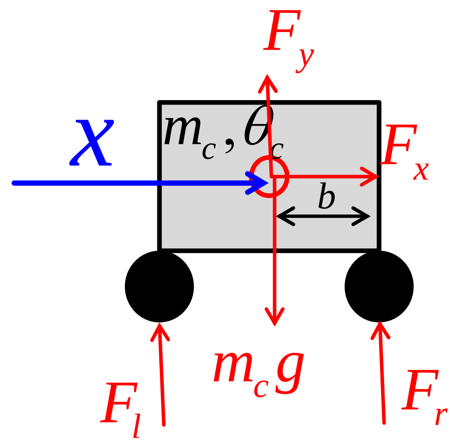
- Kinematics

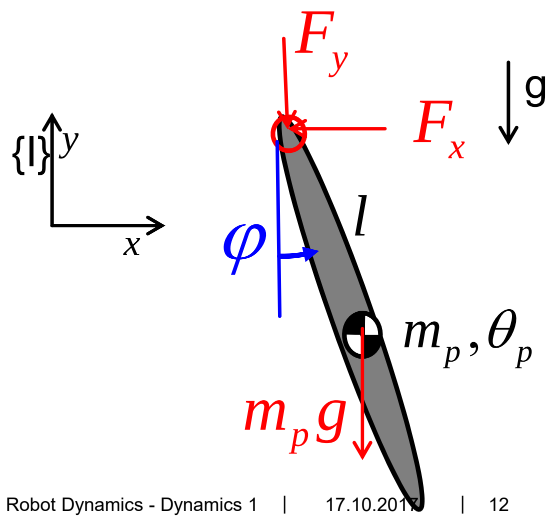
$$x_c = x \quad (7)$$
$$y_c = 0 \text{ (constraint)} \quad (8)$$
$$\varphi_c = 0 \text{ (constraint)} \quad (9)$$

$$x_p = x + l\sin(\varphi) \quad (10a) \quad \ddot{x}_p = \ddot{x} + \ddot{\varphi} l \cos(\varphi) - \dot{\varphi}^2 l \sin(\varphi) \quad (10)$$
$$y_p = -l\cos(\varphi) \quad (11a) \quad \ddot{y}_p = \ddot{\varphi} l \sin(\varphi) + \dot{\varphi}^2 l \cos(\varphi) \quad (11)$$
$$\varphi_p = \varphi \quad (12a) \quad \ddot{\varphi}_p = \ddot{\varphi} \quad (12)$$

6 equations, 6 unknowns resp.
12 equations, 12 unknowns

How many dimensions does the EoM have?



# Free Cut
## Cart pendulum

- (7),(10-12) in (1) and (4-6)

$$m_c \ddot{x} = F_x \tag{13}$$

$$m_p \left( \ddot{x} + \ddot{\varphi} l \cos(\varphi) - \dot{\varphi}^2 l \sin(\varphi) \right) = -F_x \tag{14}$$

$$m_p \left( \ddot{\varphi} l \sin(\varphi) + \dot{\varphi}^2 l \cos(\varphi) \right) = -F_y - m_p g \tag{15}$$

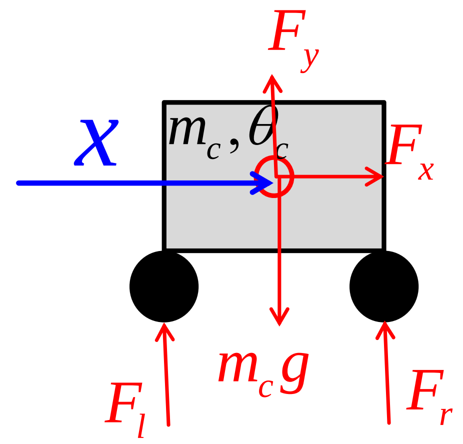
$$\theta_p \ddot{\varphi} = F_x l \cos(\varphi) + F_y l \sin(\varphi) \tag{16}$$

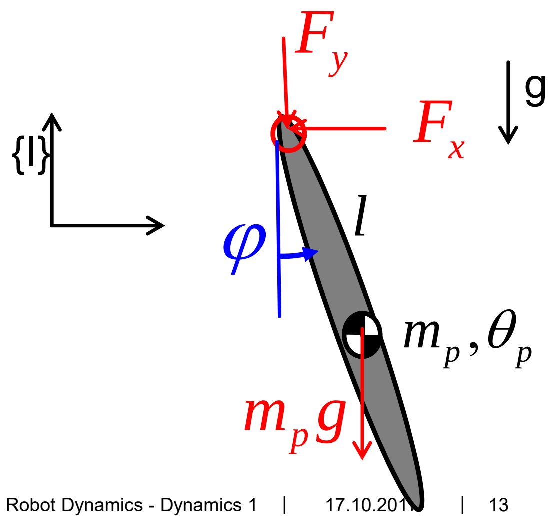
- From (13) and (14) remove $F_x$

$$\left( m_p + m_c \right) \ddot{x} + m_p l \cos(\varphi) \ddot{\varphi} - \dot{\varphi}^2 l m_p \sin(\varphi) = 0$$

- Insert (13) and (15) in (16) to remove $F_x$ and $F_y$

$$\left( \theta_p + m_p l^2 \right) \ddot{\varphi} + m_p l \cos(\varphi) \ddot{x} + g l m_p \sin(\varphi) = 0$$



# Free Cut
## Cart pendulum

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- From (13) and (14) remove $F_x$

$$\left(m_p + m_c\right)\ddot{x} + m_p l\cos(\varphi)\ddot{\varphi} - \dot{\varphi}^2 l m_p\sin(\varphi) = 0$$

- Insert (13) and (15) in (16) to remove $F_x$ and $F_y$

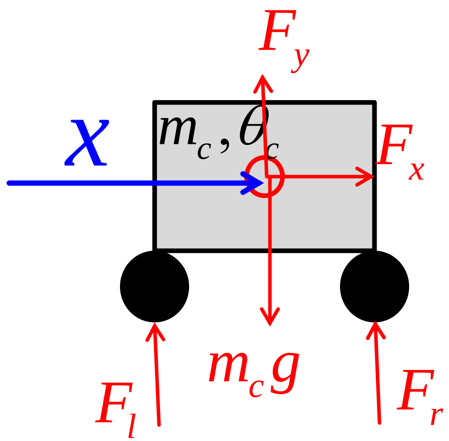
$$\left(\theta_p + m_p l^2\right)\ddot{\varphi} + m_p l\cos(\varphi)\ddot{x} + g l m_p\sin(\varphi) = 0$$

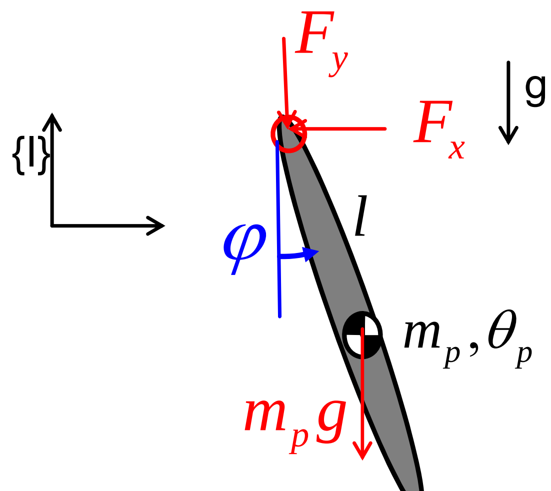
$$m_c\ddot{x}_c = F_x \tag{1}$$
$$m_c\ddot{y}_c = F_y + F_l + F_r - m_c g \tag{2}$$
$$\theta_c\ddot{\varphi}_c = F_r b - F_l b \tag{3}$$

$$m_p\ddot{x}_p = -F_x \tag{4}$$
$$m_p\ddot{y}_p = -F_y - m_p g \tag{5}$$
$$\theta_p\ddot{\varphi}_p = F_x l\cos(\varphi_p) + F_y l\sin(\varphi_p) \tag{6}$$

$$x_c = x \tag{7}$$
$$y_c = 0 \text{ (constraint)} \tag{8}$$
$$\varphi_c = 0 \text{ (constraint)} \tag{9}$$

$$\ddot{x}_p = \ddot{x} + \ddot{\varphi}l\cos(\varphi) - \dot{\varphi}^2 l\sin(\varphi) \tag{10}$$
$$\ddot{y}_p = \ddot{\varphi}l\sin(\varphi) + \dot{\varphi}^2 l\cos(\varphi) \tag{11}$$
$$\ddot{\varphi}_p = \ddot{\varphi} \tag{12}$$



# Newton-Euler in Generalized Motion Directions

- For multi-body systems $$0 = \delta W = \sum_{i=1}^{n_b} \begin{pmatrix} \delta \mathbf{r}_{S_i} \\ \delta \mathbf{\Phi}_{S_i} \end{pmatrix}^T \left( \begin{pmatrix} \dot{\mathbf{p}}_{S_i} \\ \dot{\mathbf{N}}_{S_i} \end{pmatrix} - \begin{pmatrix} \mathbf{F}_{ext,i} \\ \mathbf{T}_{ext,i} \end{pmatrix} \right) \quad \boxed{\forall \begin{pmatrix} \delta \mathbf{r}_s \\ \delta \mathbf{\Phi}_{S_i} \end{pmatrix}_{\text{consistent}}}$$
- Express the impulse/angular momentum in generalized coordinates

$$\begin{pmatrix} \mathbf{v}_s \\ \mathbf{\Omega} \end{pmatrix} = \begin{bmatrix} \mathbf{J}_P \\ \mathbf{J}_R \end{bmatrix} \dot{\mathbf{q}} \qquad \begin{pmatrix} \dot{\mathbf{p}}_{S_i} \\ \dot{\mathbf{N}}_{S_i} \end{pmatrix} = \begin{pmatrix} m \mathbf{a}_{S_i} \\ \mathbf{\Theta}_{S_i} \mathbf{\Psi}_{S_i} + \mathbf{\Omega}_{S_i} \times \mathbf{\Theta}_{S_i} \mathbf{\Omega}_{S_i} \end{pmatrix}$$

$$\begin{pmatrix} \mathbf{a}_s \\ \mathbf{\Psi} \end{pmatrix} = \begin{bmatrix} \mathbf{J}_P \\ \mathbf{J}_R \end{bmatrix} \ddot{\mathbf{q}} + \begin{bmatrix} \dot{\mathbf{J}}_P \\ \dot{\mathbf{J}}_R \end{bmatrix} \dot{\mathbf{q}} \qquad = \begin{pmatrix} m \mathbf{J}_{S_i} \\ \mathbf{\Theta}_{S_i} \mathbf{J}_{R_i} \end{pmatrix} \ddot{\mathbf{q}} + \begin{pmatrix} m \dot{\mathbf{J}}_{S_i} \dot{\mathbf{q}} \\ \mathbf{\Theta}_{S_i} \dot{\mathbf{J}}_{R_i} \dot{\mathbf{q}} + \mathbf{J}_{R_i} \dot{\mathbf{q}} \times \mathbf{\Theta}_{S_i} \mathbf{J}_{R_i} \dot{\mathbf{q}} \end{pmatrix}$$

- Virtual displacement in generalized coordinates

$$\begin{pmatrix} \delta \mathbf{r}_s \\ \delta \mathbf{\Phi} \end{pmatrix} = \begin{bmatrix} \mathbf{J}_P \\ \mathbf{J}_R \end{bmatrix} \delta \mathbf{q}$$

- With this, the principle of virtual work transforms to

$$0 = \delta \mathrm{W} = \delta \mathbf{q}^T \sum_{i=1}^{n_b} \underbrace{\begin{pmatrix} \mathbf{J}_{S_i} \\ \mathbf{J}_{R_i} \end{pmatrix}^T \begin{pmatrix} m \mathbf{J}_{S_i} \\ \mathbf{\Theta}_{S_i} \mathbf{J}_{R_i} \end{pmatrix}}_{\mathbf{M}(\mathbf{q})} \ddot{\mathbf{q}} + \begin{pmatrix} \mathbf{J}_{S_i} \\ \mathbf{J}_{R_i} \end{pmatrix}^T \underbrace{\begin{pmatrix} m \dot{\mathbf{J}}_{S_i} \dot{\mathbf{q}} \\ \mathbf{\Theta}_{S_i} \dot{\mathbf{J}}_R \dot{\mathbf{q}} + \mathbf{J}_{R_i} \dot{\mathbf{q}} \times \mathbf{\Theta}_{S_i} \mathbf{J}_{R_i} \dot{\mathbf{q}} \end{pmatrix}}_{\mathbf{b}(\mathbf{q}, \dot{\mathbf{q}})} - \underbrace{\begin{pmatrix} \mathbf{J}_{P_i} \\ \mathbf{J}_{R_i} \end{pmatrix}^T \begin{pmatrix} \mathbf{F}_{ext,i} \\ \mathbf{T}_{ext,i} \end{pmatrix}}_{\mathbf{g}(\mathbf{q})} \quad \boxed{\forall \delta \mathbf{q}}$$

# Projected Newton-Euler

- Equation of motion $\mathbf{M}(\mathbf{q})\ddot{\mathbf{q}}+\mathbf{b}(\mathbf{q},\dot{\mathbf{q}})+\mathbf{g}(\mathbf{q})=\mathbf{0}$
- Directly get the dynamic properties of a multi-body system with *n* bodies

$$\mathbf{M} = \sum_{i=1}^{n_b} \left( {}_{\mathcal{A}}\mathbf{J}_{S_i}^T \cdot m \cdot {}_{\mathcal{A}}\mathbf{J}_{S_i} + {}_{\mathcal{B}}\mathbf{J}_{R_i}^T \cdot {}_{\mathcal{B}}\mathbf{\Theta}_{S_i} \cdot {}_{\mathcal{B}}\mathbf{J}_{R_i} \right)$$

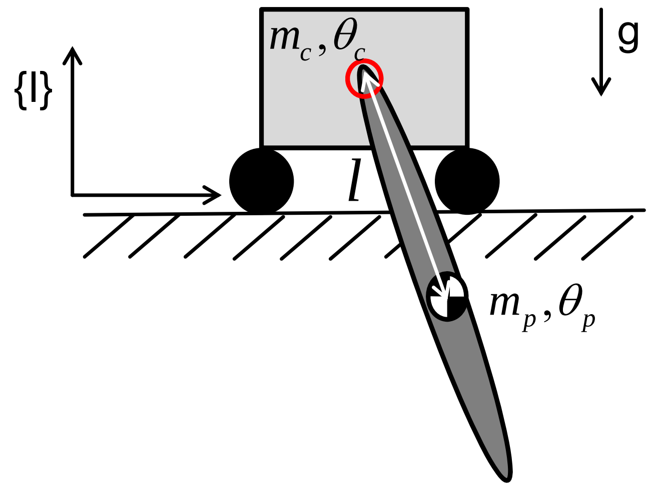
$$\mathbf{b} = \sum_{i=1}^{n_b} \left( {}_{\mathcal{A}}\mathbf{J}_{S_i}^T \cdot m \cdot {}_{\mathcal{A}}\dot{\mathbf{J}}_{S_i} \cdot \dot{\mathbf{q}} + {}_{\mathcal{B}}\mathbf{J}_{R_i}^T \cdot \left( {}_{\mathcal{B}}\mathbf{\Theta}_{S_i} \cdot {}_{\mathcal{B}}\dot{\mathbf{J}}_{R_i} \cdot \dot{\mathbf{q}} + {}_{\mathcal{B}}\mathbf{\Omega}_{S_i} \times {}_{\mathcal{B}}\mathbf{\Theta}_{S_i} \cdot {}_{\mathcal{B}}\mathbf{\Omega}_{S_i} \right) \right)$$

$$\mathbf{g} = \sum_{i=1}^{n_b} \left( -{}_{\mathcal{A}}\mathbf{J}_{S_i}^T {}_{\mathcal{A}}\mathbf{F}_{g,i} \right)$$

- For actuated systems, include actuation force as external force for each body
  - If actuators act in the direction of generalized coordinates, $\boldsymbol{\tau}$ corresponds to stacked actuator commands

# Projected Newton-Euler
## Cart pendulum example

- Find the equation of motion



# Projected Newton-Euler
## Cart pendulum example

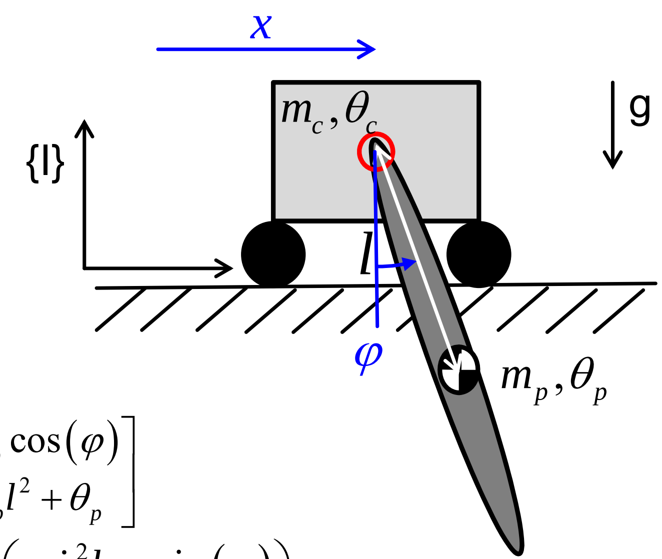
- Kinematics cart and pendulum

$$\mathbf{q} = \begin{pmatrix} x \\ \varphi \end{pmatrix} \qquad \mathbf{r}_{OS_c} = \begin{pmatrix} x \\ 0 \end{pmatrix} \qquad \mathbf{r}_{OS_p} = \begin{pmatrix} x + l\sin(\varphi) \\ -l\cos(\varphi) \end{pmatrix}$$

$$\mathbf{J}_{Pc} = \frac{d\mathbf{r}_{OS_c}}{d\mathbf{q}} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad \mathbf{J}_{Pp} = \frac{d\mathbf{r}_{OS_p}}{d\mathbf{q}} = \begin{bmatrix} 1 & l\cos(\varphi) \\ 0 & l\sin(\varphi) \end{bmatrix}$$

$$\omega_p = \dot{\varphi} \qquad \mathbf{J}_{Rp} = \frac{\partial \omega_p}{\partial \dot{\mathbf{q}}} = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

$$\dot{\mathbf{J}}_{Pc} = \frac{d\mathbf{J}_{Pc}}{dt} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad \dot{\mathbf{J}}_{Pp} = \frac{d\mathbf{J}_{Pp}}{dt} = \begin{bmatrix} 0 & -l\sin(\varphi)\dot{\varphi} \\ 0 & l\cos(\varphi)\dot{\varphi} \end{bmatrix}$$




- Equation of motion

$$\mathbf{M} = \sum \mathbf{J}_{P_i}^T m_i \mathbf{J}_{P_i} + \mathbf{J}_{R_i}^T \Theta_i \mathbf{J}_{R_i} = \mathbf{J}_{Pc}^T m_c \mathbf{J}_{Pc} + \mathbf{J}_{Pp}^T m_p \mathbf{J}_{Pp} + \mathbf{J}_{Rp}^T \theta_p \mathbf{J}_{Rp} = \begin{bmatrix} m_c + m_p & l m_p \cos(\varphi) \\ l m_p \cos(\varphi) & m_p l^2 + \theta_p \end{bmatrix}$$

$$\mathbf{b} = \sum \mathbf{J}_{P_i}^T m_i \dot{\mathbf{J}}_{P_i} \dot{\mathbf{q}} + \mathbf{J}_{R_i}^T \mathbf{\Theta}_i \dot{\mathbf{J}}_{R_i} \dot{\mathbf{q}} + \underbrace{\cancel{\left(\mathbf{J}_{R_i}\dot{\mathbf{q}}\right) \times \mathbf{\Theta}_i \mathbf{J}_{R_i} \dot{\mathbf{q}}}}_{=0 \text{ (planar system)}} = \mathbf{J}_{Pp}^T m_p \dot{\mathbf{J}}_{Pp} \dot{\mathbf{q}} = \begin{pmatrix} -\dot{\varphi}^2 l m_p \sin(\varphi) \\ 0 \end{pmatrix}$$

$$\mathbf{g} = \sum_{i=1}^{n} -\mathbf{J}_{s_i}^T \mathbf{F}_i^g = -\mathbf{J}_{P_c}^T \begin{pmatrix} 0 \\ -m_c g \end{pmatrix} - \mathbf{J}_{P_p}^T \begin{pmatrix} 0 \\ -m_p g \end{pmatrix} = \begin{pmatrix} 0 \\ m_p g l \sin(\varphi) \end{pmatrix}$$

# 3rd Method for EoM
## Lagrange II

- Lagrangian $\mathcal{L} = \underbrace{\mathcal{T}}_{\text{kinetic energy}} - \underbrace{\mathcal{U}}_{\text{potential energy}}$

- Lagrangian equation $\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}}\right) - \left(\frac{\partial \mathcal{L}}{\partial \mathbf{q}}\right) = \boldsymbol{\tau}$

- Since $\mathcal{U} = \mathcal{U}(\mathbf{q})$, $\mathcal{T} = \mathcal{T}(\mathbf{q}, \dot{\mathbf{q}})$

$$\underbrace{\boxed{\frac{d}{dt}\left(\frac{\partial \mathcal{T}}{\partial \dot{\mathbf{q}}}\right) - \frac{\partial \mathcal{T}}{\partial \mathbf{q}}}}_{\text{inertial forces}} + \underbrace{\boxed{\frac{\partial \mathcal{U}}{\partial \dot{\mathbf{q}}}}}_{\text{gravity vector}} = \boldsymbol{\tau}$$

$$\mathcal{T} = \frac{1}{2}\dot{\mathbf{q}}^T \mathbf{M} \dot{\mathbf{q}}$$

with $\frac{\partial \mathcal{T}}{\partial \dot{\mathbf{q}}} = \mathbf{M}\dot{\mathbf{q}}$

$$\mathbf{M}\ddot{\mathbf{q}} + \dot{\mathbf{M}}\dot{\mathbf{q}} - \frac{1}{2}\begin{pmatrix} \dot{\mathbf{q}}^T \frac{\partial \mathbf{M}}{\partial q_1}\dot{\mathbf{q}} \\ \vdots \\ \dot{\mathbf{q}}^T \frac{\partial \mathbf{M}}{\partial q_n}\dot{\mathbf{q}} \end{pmatrix} + \vec{g} = \mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} + \vec{b}(\mathbf{q}, \dot{\mathbf{q}}) + \vec{g}(\mathbf{q}) = \boldsymbol{\tau}$$

# Lagrange II
## Kinetic energy

- Kinetic energy in joint space $\mathcal{T} = \frac{1}{2}\dot{\mathbf{q}}^T \mathbf{M} \dot{\mathbf{q}}$
- Kinetic energy for all bodies $\mathcal{T} = \sum_{i=1}^{n_b} \left( \frac{1}{2} m_{i\,\mathcal{A}} \dot{\mathbf{r}}_{S_i}^T \, {}_{\mathcal{A}}\dot{\mathbf{r}}_{S_i} + \frac{1}{2} {}_{\mathcal{B}}\boldsymbol{\Omega}_{S_i}^T \cdot {}_{\mathcal{B}}\boldsymbol{\Theta}_{S_i} \cdot {}_{\mathcal{B}}\boldsymbol{\Omega}_{S_i} \right)$
- From kinematics we know that

$$\dot{\mathbf{r}}_{S_i} = \mathbf{J}_{S_i} \dot{\mathbf{q}}$$
$$\boldsymbol{\Omega}_{S_i} = \mathbf{J}_{R_i} \dot{\mathbf{q}}$$

- Hence we get

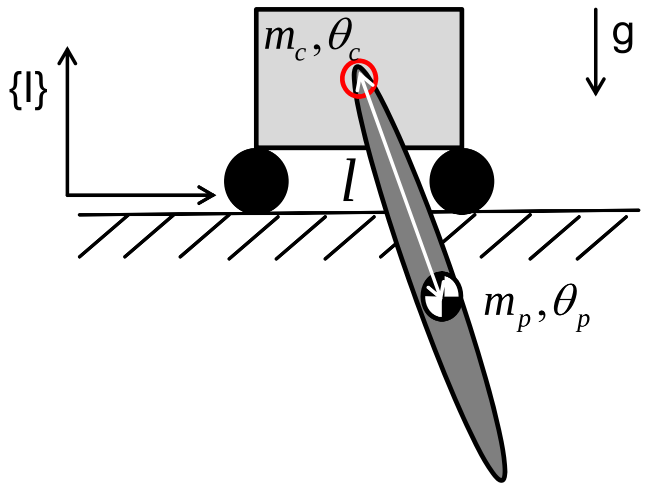
$$\mathcal{T}(\mathbf{q}, \dot{\mathbf{q}}) = \frac{1}{2} \dot{\mathbf{q}}^T \underbrace{\left( \sum_{i=1}^{n_b} \left( \mathbf{J}_{S_i}^T m \mathbf{J}_{S_i} + \mathbf{J}_{R_i}^T \boldsymbol{\Theta}_{S_i} \mathbf{J}_{R_i} \right) \right)}_{\mathbf{M}(\mathbf{q})} \dot{\mathbf{q}}$$

# Lagrange II
## Potential energy

- Two sources for potential forces
  - Gravitational forces $\mathbf{F}_{g_i} = m_i \, g \, {}_I\mathbf{e}_g$ $\longrightarrow$ $\mathcal{U}_g = -\sum_{i=1}^{n_b} \mathbf{r}_{S_i}^T \, \mathbf{F}_{g_i}$
  - Spring forces $\mathbf{F}_E = k_j \left( \left\| \mathbf{r} - \mathbf{r}_0 \right\| - d_0 \right) \frac{\mathbf{r} - \mathbf{r}_0}{\left\| \mathbf{r} - \mathbf{r}_0 \right\|}$ $\longrightarrow$ $\mathcal{U}_{E_j} = \frac{1}{2} k_j \left( d(\mathbf{q}) - d_0 \right)^2$

# Lagrange II
## Cart pendulum example

- Find the equation of motion



# Lagrange II
## Cart pendulum example

- Kinematics cart and pendulum

$$\mathbf{r}_{OS_c} = \begin{pmatrix} x \\ 0 \end{pmatrix} \qquad \mathbf{r}_{OS_p} = \begin{pmatrix} x + l\sin(\varphi) \\ -l\cos(\varphi) \end{pmatrix} \qquad \varphi_p = \varphi$$

$$\dot{\mathbf{r}}_{S_c} = \begin{pmatrix} \dot{x} \\ 0 \end{pmatrix} \qquad \dot{\mathbf{r}}_{S_p} = \begin{pmatrix} \dot{x} + l\cos(\varphi)\dot{\varphi} \\ l\sin(\varphi)\dot{\varphi} \end{pmatrix} \qquad \omega_p = \dot{\varphi}$$

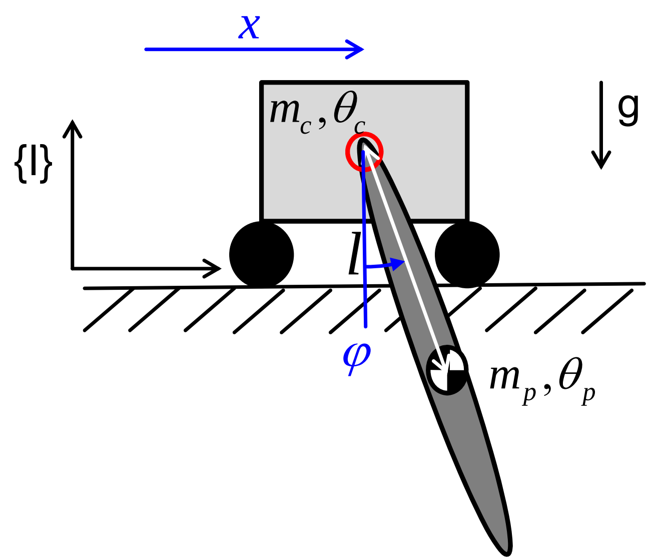
- Kinetic and potential energy

$$\mathcal{T} = \sum \frac{1}{2}\dot{\mathbf{r}}_{S_i}^T m_i \dot{\mathbf{r}}_{S_i} + \frac{1}{2}\boldsymbol{\omega}_i^T \Theta_i \boldsymbol{\omega}_i = \frac{1}{2}m_c\dot{x}^2 + \frac{1}{2}m_p\dot{x}^2 + \frac{1}{2}m_p l^2\dot{\varphi}^2 + m_p\dot{x}l\cos(\varphi)\dot{\varphi} + \frac{1}{2}\theta\dot{\varphi}^2$$

$$\mathcal{U} = -m_p g l\cos(\varphi)$$

0-level can be chosen

- Equation of motion




$$\frac{\partial \mathcal{T}}{\partial \dot{\mathbf{q}}} = \begin{pmatrix} m_c\dot{x} + m_p\dot{x} + m_p l\cos(\varphi)\dot{\varphi} \\ m_p l^2\dot{\varphi} + m_p\dot{x}l\cos(\varphi) + \theta\dot{\varphi} \end{pmatrix}$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{T}}{\partial \dot{\mathbf{q}}}\right) - \frac{\partial \mathcal{T}}{\partial \mathbf{q}} + \frac{\partial \mathcal{U}}{\partial \mathbf{q}} = 0 \qquad \frac{d}{dt}\left(\frac{\partial \mathcal{T}}{\partial \dot{\mathbf{q}}}\right) = \begin{pmatrix} m_c\ddot{x} + m_p\ddot{x} + m_p l\cos(\varphi)\ddot{\varphi} - m_p l\sin(\varphi)\dot{\varphi}^2 \\ m_p l^2\ddot{\varphi} + m_p\ddot{x}l\cos(\varphi) - m_p\dot{x}l\sin(\varphi)\dot{\varphi} + \theta\ddot{\varphi} \end{pmatrix}$$

$$-\frac{\partial \mathcal{T}}{\partial \mathbf{q}} = \begin{pmatrix} 0 \\ m_p\dot{x}l\sin(\varphi)\dot{\varphi} \end{pmatrix}$$

$$\frac{\partial \mathcal{U}}{\partial \mathbf{q}} = \begin{pmatrix} 0 \\ m_p g l\sin(\varphi) \end{pmatrix}$$

$$\begin{pmatrix} m_c\ddot{x} + m_p\ddot{x} + m_p l\cos(\varphi)\ddot{\varphi} - m_p l\sin(\varphi)\dot{\varphi}^2 \\ m_p l^2\ddot{\varphi} + m_p\ddot{x}l\cos(\varphi) + \theta\ddot{\varphi} + m_p g l\sin(\varphi) \end{pmatrix} = 0$$

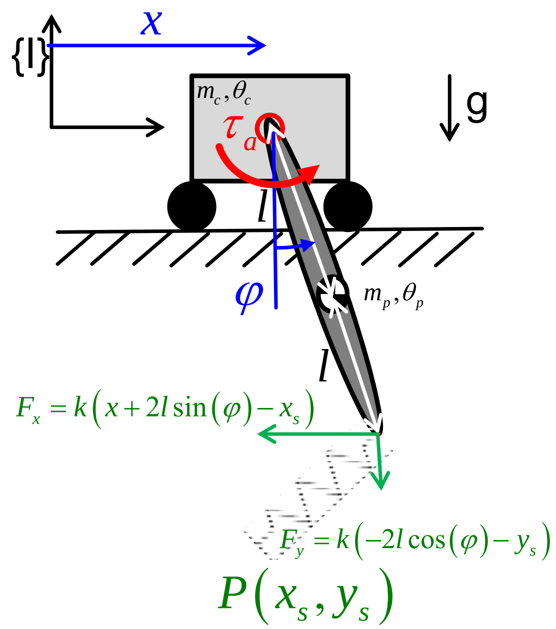
# External Forces

- Given: $n_{f,ext}$ external forces $\mathbf{F}_j$
- Generalized forces are calculated as: $\boldsymbol{\tau}_{F,ext} = \sum_{j=1}^{n_{f,ext}} \mathbf{J}_{P,j}{}^T \mathbf{F}_j$
- Given: $n_{m,ext}$ external torques $\mathbf{T}_k$
- Generalized forces are calculated $\boldsymbol{\tau}_{T,ext} = \sum_{k=1}^{n_{m,ext}} \mathbf{J}_{R,k}{}^T \mathbf{T}_{ext,k}$
- For actuator torques: $\boldsymbol{\tau}_{a,k} = \left(\mathbf{J}_{S_k} - \mathbf{J}_{S_{k-1}}\right)^T \mathbf{F}_{a,k} + \left(\mathbf{J}_{R_k} - \mathbf{J}_{R_{k-1}}\right)^T \mathbf{T}_{a,k}$

# External Forces
## Cart pendulum example

- Equation of motion without actuation

$$\underbrace{\begin{bmatrix} m_c + m_p & l m_p \cos(\varphi) \\ l m_p \cos(\varphi) & m_p l^2 + \theta_p \end{bmatrix}}_{\mathbf{M}} \ddot{\mathbf{q}} + \underbrace{\begin{pmatrix} -\dot{\varphi}^2 l m_p \sin(\varphi) \\ 0 \end{pmatrix}}_{\mathbf{b}} + \underbrace{\begin{pmatrix} 0 \\ m_p g l \sin(\varphi) \end{pmatrix}}_{\mathbf{g}} = \mathbf{0}$$




- Add actuator for the pendulum $\tau_a$
  - Action on pendulum
  - Reaction on cart

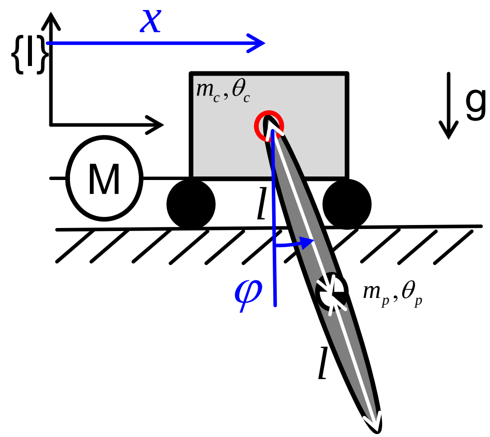
$$\left.\begin{matrix} T_p = \tau_a & \mathbf{J}_{Rp} = [0 \quad 1] \\ T_c = -\tau_a & \mathbf{J}_{Rc} = [0 \quad 0] \end{matrix}\right\} \boldsymbol{\tau} = \sum \mathbf{J}_{R,i}^T \mathbf{T}_i = \mathbf{J}_{Rc}^T \mathbf{T}_c + \mathbf{J}_{Rp}^T \mathbf{T}_p = \begin{pmatrix} 0 \\ \tau_a \end{pmatrix}$$

- Add spring to the pendulum
  - (world attachment point P, zero length 0, stiffness k)
  - Action on pendulum

$$\mathbf{F}_s = \begin{pmatrix} -F_x \\ -F_y \end{pmatrix} \quad \left.\begin{matrix} \mathbf{r} = \begin{pmatrix} x + 2l\sin(\varphi) \\ -2l\cos(\varphi) \end{pmatrix} \\ \mathbf{J}_s = \dfrac{\partial \mathbf{r}_s}{\partial \mathbf{q}} = \begin{bmatrix} 1 & 2l\cos(\varphi) \\ 0 & 2l\sin(\varphi) \end{bmatrix} \end{matrix}\right\} \boldsymbol{\tau} = \mathbf{J}^T \mathbf{F}_s = \begin{pmatrix} -F_x \\ -2l\left(F_x \cos(\varphi) + F_y \sin(\varphi)\right) \end{pmatrix}$$

# External Forces
## Cart pendulum example

- What is the external force coming from the motor



# External Forces
## Cart pendulum example

- What is the external force coming from the motor
  - Action on cart $F_{act}$

$$F_c = F_{act} \qquad \mathbf{J}_{Pc} = \begin{bmatrix} 1 & 0 \end{bmatrix} \qquad \boldsymbol{\tau} = \mathbf{J}^T F_c = \begin{pmatrix} F_{act} \\ 0 \end{pmatrix}$$