Lecture «Robot Dynamics»: Quaternion JPL/Hamiltonian

151-0851-00 V

lecture:	CAB G11	Tuesday 10:15 – 12:00, every week
exercise:	HG E1.2	Wednesday 8:15 – 10:00, according to schedule (about every 2nd week)

Marco Hutter, Roland Siegwart, and Thomas Stastny

Difference Hamiltonian and JPL convention

Table 1: Hamilton vs. JPL quaternion conventions with respect to the 4 binary choices

	Quaternion type	Hamilton	JPL
1	Components order	$(q_w\ ,\ \mathbf{q}_v)$	$(\mathbf{q}_v\ ,\ q_w)$
2	Algebra	$ij = k$	$ij = -k$
	Handedness	Right-handed	Left-handed
3	Function	Passive	Passive
4	Right-to-left products mean	Local-to-Global	Global-to-Local
	Default notation, $\mathbf{q}$	$\mathbf{q} \triangleq \mathbf{q}_{\mathcal{GL}}$	$\mathbf{q} \triangleq \mathbf{q}_{\mathcal{LG}}$
	Default operation	$\mathbf{x}_{\mathcal{G}} = \mathbf{q} \otimes \mathbf{x}_{\mathcal{L}} \otimes \mathbf{q}^*$	$\mathbf{x}_{\mathcal{L}} = \mathbf{q} \otimes \mathbf{x}_{\mathcal{G}} \otimes \mathbf{q}^*$

$$\begin{aligned} p\left({}_I r\right) &= \zeta_{IB} \otimes p\left({}_B r\right) \otimes \zeta_{IB}^T \\ &= \mathbf{M}_l^h\left(\zeta_{IB}\right) \mathbf{M}_r^h\left(\zeta_{IB}^T\right) p\left({}_B r\right) \end{aligned}$$

$$\begin{aligned} p\left({}_B r\right) &= \zeta_{BI} \otimes p\left({}_I r\right) \otimes \zeta_{BI}^T \\ &= \mathbf{M}_l^J\left(\zeta_{BI}\right) \mathbf{M}_r^J\left(\zeta_{BI}^T\right) p\left({}_I r\right) \end{aligned}$$

Hamiltonian
Algebra

- Product of quaternions
  - Given two quaternions **q** and **p**, the product is defined as

$$
\begin{aligned}
\boldsymbol{\zeta} \otimes \boldsymbol{p} &= (q_0 + q_1\boldsymbol{i} + q_2\boldsymbol{j} + q_3\boldsymbol{k})(p_0 + p_1\boldsymbol{i} + p_2\boldsymbol{j} + p_3\boldsymbol{k}) \\
&= q_0p_0 + q_0p_1\boldsymbol{i} + q_0p_2\boldsymbol{j} + q_0p_3\boldsymbol{k} \\
&\quad + q_1p_0\boldsymbol{i} + q_1p_1\boldsymbol{ii} + q_1p_2\boldsymbol{ij} + q_1p_3\boldsymbol{ik} \\
&\quad + q_2p_0\boldsymbol{j} + q_2p_1\boldsymbol{ji} + q_2p_2\boldsymbol{jj} + q_2p_3\boldsymbol{jk} \\
&\quad + q_3p_0\boldsymbol{k} + q_3p_1\boldsymbol{ki} + q_3p_2\boldsymbol{kj} + q_3p_3\boldsymbol{kk} \\
&= q_0p_0 - q_1p_1 - q_2p_2 - q_3p_3 \\
&\quad + (q_0p_1 + q_1p_0 + q_2p_3 - q_3p_2)\boldsymbol{i} \\
&\quad + (q_0p_2 - q_1p_3 + q_2p_0 + q_3p_1)\boldsymbol{j} \\
&\quad + (q_0p_3 + q_1p_2 - q_2p_1 + q_3p_0)\boldsymbol{k} \\
&= \begin{bmatrix} q_0 & -q_1 & -q_2 & -q_3 \\ q_1 & q_0 & -q_3 & q_2 \\ q_2 & q_3 & q_0 & -q_1 \\ q_3 & -q_2 & q_1 & q_0 \end{bmatrix} \begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{pmatrix} = \underbrace{\begin{bmatrix} q_0 & -\breve{\boldsymbol{\zeta}}^T \\ \breve{\boldsymbol{\zeta}} & q_0\boldsymbol{I} + [\breve{\boldsymbol{\zeta}}]_\times \end{bmatrix}}_{=:\boldsymbol{M}_l^{\boldsymbol{h}}(\boldsymbol{\zeta})} \boldsymbol{p} = \boldsymbol{M}_l^{\boldsymbol{h}}(\boldsymbol{\zeta})\boldsymbol{p} \\
&= \begin{bmatrix} p_0 & -p_1 & -p_2 & -p_3 \\ p_1 & p_0 & p_3 & -p_2 \\ p_2 & -p_3 & p_0 & p_1 \\ p_3 & p_2 & -p_1 & p_0 \end{bmatrix} \begin{pmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{pmatrix} = \underbrace{\begin{bmatrix} p_0 & -\breve{\boldsymbol{p}}^T \\ \breve{\boldsymbol{p}} & p_0\boldsymbol{I} - [\breve{\boldsymbol{p}}]_\times \end{bmatrix}}_{=:\boldsymbol{M}_r^{\boldsymbol{h}}(\boldsymbol{p})} \boldsymbol{\zeta} = \boldsymbol{M}_r^{\boldsymbol{h}}(\boldsymbol{p})\boldsymbol{\zeta}
\end{aligned}
$$

Hamiltonian convention

$$\xi = \xi_0 + \xi_1 i + \xi_2 j + \xi_3 k$$

$$i^2 = j^2 = k^2 = ijk = -1$$

$$ij = -ji = -ijk^2 = k$$

$$jk = -kj = i$$

$$ki = -ik = j$$

ETH zürich

Hamiltonian

Derivation of rotation matrix

- Derivation of rotation matrix ($\boldsymbol{\zeta} = \boldsymbol{\zeta}_{BI}$):
  - $\boldsymbol{p}({}_B\boldsymbol{r}) = \boldsymbol{\zeta} \otimes \boldsymbol{p}({}_I\boldsymbol{r}) \otimes \boldsymbol{\zeta}^T = \boldsymbol{M_l}(\boldsymbol{\zeta})\boldsymbol{M_r}(\boldsymbol{\zeta}^T)\begin{pmatrix} 0 \\ {}_I\boldsymbol{r} \end{pmatrix}$
  - $\begin{pmatrix} 0 \\ {}_B\boldsymbol{r} \end{pmatrix} = \begin{bmatrix} \zeta_0 & -\check{\boldsymbol{\zeta}}^T \\ \check{\boldsymbol{\zeta}} & \zeta_0\boldsymbol{I} + [\check{\zeta}]_\times \end{bmatrix}\begin{bmatrix} \zeta_0 & \check{\boldsymbol{\zeta}}^T \\ -\check{\boldsymbol{\zeta}} & \zeta_0\boldsymbol{I} + [\check{\zeta}]_\times \end{bmatrix}\begin{pmatrix} 0 \\ {}_I\boldsymbol{r} \end{pmatrix}$
  - $\begin{pmatrix} 0 \\ {}_B\boldsymbol{r} \end{pmatrix} = \begin{bmatrix} \zeta_0^2 + |\check{\zeta}|^2 & \zeta_0\check{\boldsymbol{\zeta}}^T - \zeta_0\check{\boldsymbol{\zeta}}^T - \check{\boldsymbol{\zeta}}^T[\check{\zeta}]_\times \\ \zeta_0\check{\boldsymbol{\zeta}} - \zeta_0\check{\boldsymbol{\zeta}} - [\check{\zeta}]_\times\check{\boldsymbol{\zeta}} & \check{\boldsymbol{\zeta}}\check{\boldsymbol{\zeta}}^T + \zeta_0^2\boldsymbol{I} + 2\zeta_0[\check{\zeta}]_\times + \underline{[\check{\zeta}]_\times[\check{\zeta}]_\times} \end{bmatrix}\begin{pmatrix} 0 \\ {}_I\boldsymbol{r} \end{pmatrix}$
  - $\begin{pmatrix} 0 \\ {}_B\boldsymbol{r} \end{pmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \left(\zeta_0^2 - |\check{\zeta}|^2\right)\boldsymbol{I} + 2\zeta_0[\check{\zeta}]_\times + \boldsymbol{2}\check{\boldsymbol{\zeta}}\check{\boldsymbol{\zeta}}^T \end{bmatrix}\begin{pmatrix} 0 \\ {}_I\boldsymbol{r} \end{pmatrix}$
  - ${}_B\boldsymbol{r} = \left(\left(\zeta_0^2 - |\check{\zeta}|^2\right)\boldsymbol{I} + 2\zeta_0[\check{\zeta}]_\times + \boldsymbol{2}\check{\boldsymbol{\zeta}}\check{\boldsymbol{\zeta}}^T\right)\,{}_I\boldsymbol{r}$
  - $\boldsymbol{C_{BI}}(\boldsymbol{\zeta}) = (2\zeta_0^2 - 1)\boldsymbol{I} + 2\zeta_0[\check{\zeta}]_\times + \boldsymbol{2}\check{\boldsymbol{\zeta}}\check{\boldsymbol{\zeta}}^T$

$$\boxed{[\check{\zeta}]_\times[\check{\zeta}]_\times = \check{\zeta}\check{\zeta}^T - |\check{\zeta}|^2\mathbf{I}}$$

$$\boxed{\begin{aligned} \boldsymbol{M_r}(\boldsymbol{\zeta}) &= \begin{bmatrix} \zeta_0 & -\check{\boldsymbol{\zeta}}^T \\ \check{\boldsymbol{\zeta}} & \zeta_0\boldsymbol{I} - [\check{\zeta}]_\times \end{bmatrix} \\ [\check{\zeta}^T]_\times &= -[\check{\zeta}]_\times \\ \boldsymbol{\zeta}^{-1} &= \boldsymbol{\zeta}^T = \begin{pmatrix} \zeta_0 \\ -\check{\boldsymbol{\zeta}} \end{pmatrix} \end{aligned}}$$

JPL
algebra

- Product of quaternions
  - Given two quaternions **q** and **p**, the product is defined as

$$
\begin{aligned}
\boldsymbol{\zeta} \otimes \boldsymbol{p} &= (q_0 + q_1\boldsymbol{i} + q_2\boldsymbol{j} + q_3\boldsymbol{k})(p_0 + p_1\boldsymbol{i} + p_2\boldsymbol{j} + p_3\boldsymbol{k}) \\
&= q_0p_0 + q_0p_1\boldsymbol{i} + q_0p_2\boldsymbol{j} + q_0p_3\boldsymbol{k} \\
&\quad + q_1p_0\boldsymbol{i} + q_1p_1\boldsymbol{ii} + q_1p_2\boldsymbol{ij} + q_1p_3\boldsymbol{ik} \\
&\quad + q_2p_0\boldsymbol{j} + q_2p_1\boldsymbol{ji} + q_2p_2\boldsymbol{jj} + q_2p_3\boldsymbol{jk} \\
&\quad + q_3p_0\boldsymbol{k} + q_3p_1\boldsymbol{ki} + q_3p_2\boldsymbol{kj} + q_3p_3\boldsymbol{kk} \\
&= q_0p_0 - q_1p_1 - q_2p_2 - q_3p_3 \\
&\quad + (q_0p_1 + q_1p_0 - q_2p_3 + q_3p_2)\boldsymbol{i} \\
&\quad + (q_0p_2 + q_1p_3 + q_2p_0 - q_3p_1)\boldsymbol{j} \\
&\quad + (q_0p_3 - q_1p_2 + q_2p_1 + q_3p_0)\boldsymbol{k} \\
&= \begin{bmatrix} q_0 & -q_1 & -q_2 & -q_3 \\ q_1 & q_0 & q_3 & -q_2 \\ q_2 & -q_3 & q_0 & q_1 \\ q_3 & q_2 & -q_1 & q_0 \end{bmatrix} \begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{pmatrix} = \underbrace{\begin{bmatrix} q_0 & -\check{\boldsymbol{\zeta}}^T \\ \check{\boldsymbol{\zeta}} & q_0\boldsymbol{I} \boxed{-} \check{\boldsymbol{\zeta}}^{\times} \end{bmatrix}}_{=:\boldsymbol{M}_l^J(\boldsymbol{\zeta})} \boldsymbol{p} = \boldsymbol{M}_l^J(\boldsymbol{\zeta})\boldsymbol{p} \\
&= \begin{bmatrix} p_0 & -p_1 & -p_2 & -p_3 \\ p_1 & p_0 & -p_3 & p_2 \\ p_2 & p_3 & p_0 & -p_1 \\ p_3 & -p_2 & p_1 & p_0 \end{bmatrix} \begin{pmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{pmatrix} = \underbrace{\begin{bmatrix} p_0 & -\check{\boldsymbol{p}}^T \\ \check{\boldsymbol{p}} & p_0\boldsymbol{I} \boxed{+} \check{\boldsymbol{p}}^{\times} \end{bmatrix}}_{=:\boldsymbol{M}_r^J(\boldsymbol{p})} \boldsymbol{\zeta} = \boldsymbol{M}_r^J(\boldsymbol{p})\boldsymbol{\zeta}
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{ji} = -\boldsymbol{k}^2\boldsymbol{ji} &= \boldsymbol{k} \\
\boldsymbol{ik} &= \boldsymbol{j} \\
\boldsymbol{ij} &= -\boldsymbol{k} \\
\boldsymbol{jk} &= -\boldsymbol{i} \\
\boldsymbol{ki} &= -\boldsymbol{j} \\
\boldsymbol{kj} &= \boldsymbol{i}
\end{aligned}
$$

!!!! JPL !!!

JPL
Derivation of rotation matrix

- Derivation of rotation matrix ($\boldsymbol{\zeta} = \boldsymbol{\zeta}_{IB}^{J}$):
  - $\boldsymbol{p}({}_I\boldsymbol{r}) = \boldsymbol{\zeta} \otimes \boldsymbol{p}({}_I\boldsymbol{r}) \otimes \boldsymbol{\zeta}^T = \boldsymbol{M}_l(\boldsymbol{\zeta})\boldsymbol{M}_r(\boldsymbol{\zeta}^T)\begin{pmatrix} 0 \\ {}_B\boldsymbol{r} \end{pmatrix}$
  - $\begin{pmatrix} 0 \\ {}_I\boldsymbol{r} \end{pmatrix} = \begin{bmatrix} q_0 & -\check{\boldsymbol{\zeta}}^T \\ \check{\boldsymbol{\zeta}} & q_0\boldsymbol{I} - \check{\boldsymbol{\zeta}}^\times \end{bmatrix}\begin{bmatrix} q_0 & \check{\boldsymbol{\zeta}}^T \\ -\check{\boldsymbol{\zeta}} & q_0\boldsymbol{I} - \check{\boldsymbol{\zeta}}^\times \end{bmatrix}\begin{pmatrix} 0 \\ {}_B\boldsymbol{r} \end{pmatrix}$
  - $\begin{pmatrix} 0 \\ {}_I\boldsymbol{r} \end{pmatrix} = \begin{bmatrix} q_0^2 + \left|\check{\zeta}\right|^2 & q_0\check{\boldsymbol{\zeta}}^T - q_0\check{\boldsymbol{\zeta}}^T + \check{\boldsymbol{\zeta}}^T\check{\boldsymbol{\zeta}}^\times \\ q_0\check{\boldsymbol{\zeta}} - q_0\check{\boldsymbol{\zeta}} + \check{\boldsymbol{\zeta}}^\times\check{\boldsymbol{\zeta}} & \check{\boldsymbol{\zeta}}\check{\boldsymbol{\zeta}}^T + q_0^2\boldsymbol{I} - 2q_0\check{\boldsymbol{\zeta}}^\times + \check{\boldsymbol{\zeta}}^\times\check{\boldsymbol{\zeta}}^\times \end{bmatrix}\begin{pmatrix} 0 \\ {}_B\boldsymbol{r} \end{pmatrix}$
  - $\begin{pmatrix} 0 \\ {}_I\boldsymbol{r} \end{pmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \left(q_0^2 - \left|\check{\zeta}\right|^2\right)\boldsymbol{I} - 2q_0\check{\boldsymbol{\zeta}}^\times + \boldsymbol{2}\check{\boldsymbol{\zeta}}\check{\boldsymbol{\zeta}}^T \end{bmatrix}\begin{pmatrix} 0 \\ {}_B\boldsymbol{r} \end{pmatrix}$
  - ${}_I\boldsymbol{r} = \left(\left(q_0^2 - \left|\check{\zeta}\right|^2\right)\boldsymbol{I} - 2q_0\check{\boldsymbol{\zeta}}^\times + \boldsymbol{2}\check{\boldsymbol{\zeta}}\check{\boldsymbol{\zeta}}^T\right)\,{}_I\boldsymbol{r}$
  - $\boldsymbol{C_{IB}}(\boldsymbol{\zeta}) = (2q_0^2 - 1)\boldsymbol{I} - 2q_0\check{\boldsymbol{\zeta}}^\times + \boldsymbol{2}\check{\boldsymbol{\zeta}}\check{\boldsymbol{\zeta}}^T$
  - $\boldsymbol{C_{BI}}(\boldsymbol{\zeta}) = \boldsymbol{C}_{\boldsymbol{BI}}^T(\boldsymbol{\zeta}) = (2q_0^2 - 1)\boldsymbol{I} + 2q_0\check{\boldsymbol{\zeta}}^\times + \boldsymbol{2}\check{\boldsymbol{\zeta}}\check{\boldsymbol{\zeta}}^T$