



Lecture «Robot Dynamics»: Intro to Dynamics

151-0851-00 V

lecture:	CAB G11	Tuesday 10:15 – 12:00, every week
exercise:	HG E1.2	Wednesday 8:15 – 10:00, according to schedule (about every 2nd week)

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19.09.2017	Intro and Outline	Course Introduction; Recapitulation Position, Linear Velocity			
26.09.2017	Kinematics 1	Rotation and Angular Velocity; Rigid Body Formulation, Transformation	26.09.2017	Exercise 1a	Kinematics Modeling the ABB arm
03.10.2017	Kinematics 2	Kinematics of Systems of Bodies; Jacobians	03.10.2017	Exercise 1b	Differential Kinematics of the ABB arm
10.10.2017	Kinematics 3	Kinematic Control Methods: Inverse Differential Kinematics, Inverse Kinematics; Rotation Error; Multi-task Control	10.10.2017	Exercise 1c	Kinematic Control of the ABB Arm
17.10.2017	Dynamics L1	Multi-body Dynamics	17.10.2017	Exercise 2a	Dynamic Modeling of the ABB Arm
24.10.2017	Dynamics L2	Floating Base Dynamics	24.10.2017		
31.10.2017	Dynamics L3	Dynamic Model Based Control Methods	31.10.2017	Exercise 2b	Dynamic Control Methods Applied to the ABB arm
07.11.2017	Legged Robot	Dynamic Modeling of Legged Robots & Control	07.11.2017	Exercise 3	Legged robot
14.11.2017	Case Studies 1	Legged Robotics Case Study	14.11.2017		
21.11.2017	Rotorcraft	Dynamic Modeling of Rotorcraft & Control	21.11.2017	Exercise 4	Modeling and Control of Multicopter
28.11.2017	Case Studies 2	Rotor Craft Case Study	28.11.2017		
05.12.2017	Fixed-wing	Dynamic Modeling of Fixed-wing & Control	05.12.2017	Exercise 5	Fixed-wing Control and Simulation
12.12.2017	Case Studies 3	Fixed-wing Case Study (Solar-powered UAVs - AtlantikSolar, Vertical Take-off and Landing UAVs – Wingtra)			
19.12.2017	Summery and Outlook	Summery; Wrap-up; Exam			

Recapitulation of Kinematics

- Kinematics = **description** of motions
 - Translations and rotations
 - Various representations (Euler, quaternions, etc.)
 - Instantaneous/Differential kinematics
 - Jacobians and geometric Jacobians
 - Inverse kinematics and control
 - Floating base systems (unactuated base and contacts)

$$\chi_e = \chi_e(\mathbf{q})$$

$$\mathbf{w}_e = \begin{pmatrix} \mathbf{v}_e \\ \boldsymbol{\omega}_e \end{pmatrix} = \mathbf{J}_{e0}(\mathbf{q}) \dot{\mathbf{q}}$$

Dynamics in Robotics



Dynamics in Robotics



Dynamics

Outline

- Description of “cause of motion”
 - Input τ Force/Torque acting on system
 - Output $\ddot{\mathbf{q}}$ Motion of the system
- Principle of virtual work
 - Newton’s law for particles
 - Conservation of impulse and angular momentum
- 3 methods to get the EoM
 - Newton-Euler: Free cut and conservation of impulse & angular momentum for each body
 - Projected Newton-Euler (generalized coordinates)
 - Lagrange II (energy)
- Introduction to dynamics of floating base systems
 - External forces

$$\mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{b}(\mathbf{q}, \dot{\mathbf{q}}) + \mathbf{g}(\mathbf{q}) = \boldsymbol{\tau} + \mathbf{J}_c^T \mathbf{F}_c$$

$\ddot{\mathbf{q}}$	Generalized coordinates
$\mathbf{M}(\mathbf{q})$	Mass matrix
$\mathbf{b}(\mathbf{q}, \dot{\mathbf{q}})$	Centrifugal and Coriolis forces
$\mathbf{g}(\mathbf{q})$	Gravity forces
$\boldsymbol{\tau}$	Generalized forces
\mathbf{F}_c	External forces
\mathbf{J}_c	Contact Jacobian

Principle of Virtual Work

- Principle of virtual work (D'Alembert's Principle)

- Dynamic equilibrium imposes zero virtual work

$$\delta W = \int_{\mathcal{B}} \delta \mathbf{r}^T \cdot (\ddot{\mathbf{r}} dm - d\mathbf{F}_{ext}) = 0$$

variational parameter

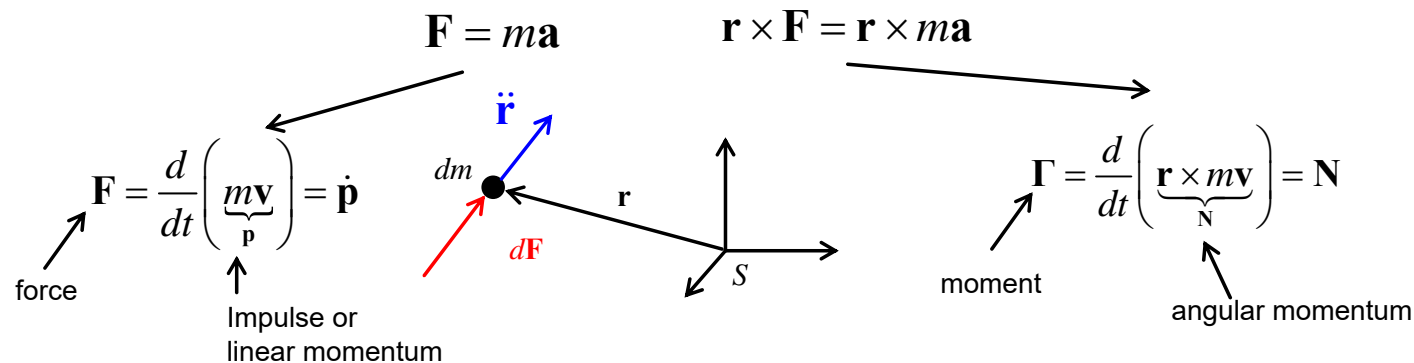
$d\mathbf{F}_{ext}$ external forces acting on element i

$\ddot{\mathbf{r}}$ acceleration of element i

dm mass of element i

$\delta \mathbf{r}$ virtual displacement of element i

- Newton's law for every particle in direction it can move



Virtual Displacements of Single Rigid Bodies

■ Rigid body Kinematics

$$\mathbf{r} = \mathbf{r}_{OS} + \boldsymbol{\rho}$$

$$\dot{\mathbf{r}} = \mathbf{v}_S + \boldsymbol{\Omega} \times \boldsymbol{\rho} = \begin{bmatrix} \mathbb{I}_{3 \times 3} & -[\boldsymbol{\rho}]_{\times} \end{bmatrix} \begin{pmatrix} \mathbf{v}_S \\ \boldsymbol{\Omega} \end{pmatrix}$$

$$\ddot{\mathbf{r}} = \mathbf{a}_S + \dot{\boldsymbol{\Psi}} \times \boldsymbol{\rho} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{\rho}) = \begin{bmatrix} \mathbb{I}_{3 \times 3} & -[\boldsymbol{\rho}]_{\times} \end{bmatrix} \begin{pmatrix} \mathbf{a}_S \\ \dot{\boldsymbol{\Psi}} \end{pmatrix} + [\boldsymbol{\Omega}]_{\times} [\boldsymbol{\Omega}]_{\times} \boldsymbol{\rho}$$

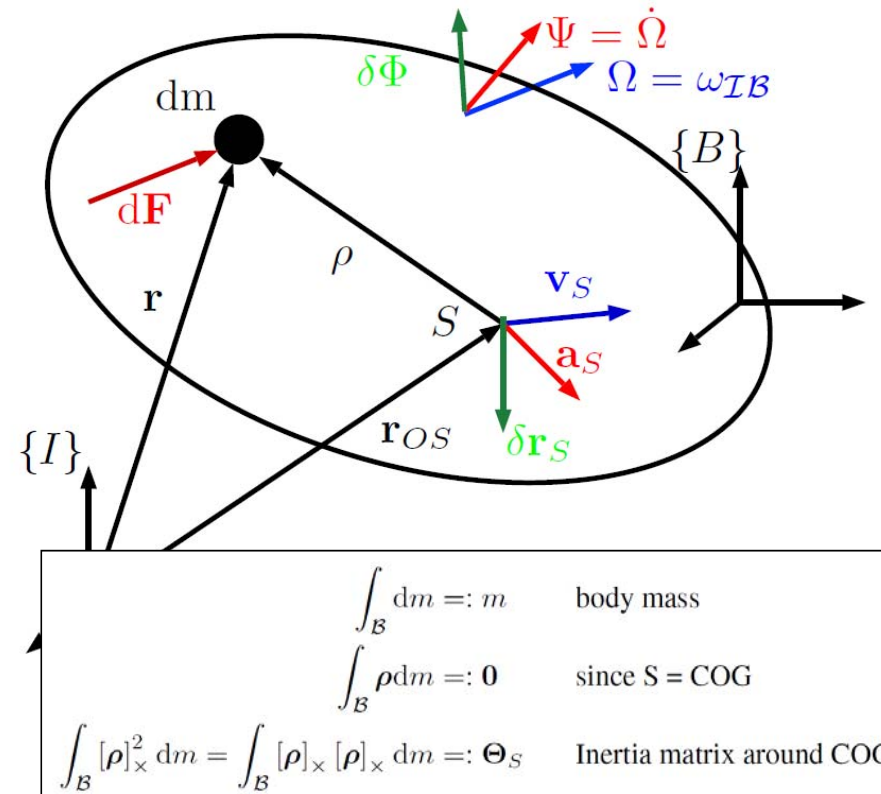
$$\delta \mathbf{r} = \delta \mathbf{r}_S + \delta \boldsymbol{\Phi} \times \boldsymbol{\rho} = \begin{bmatrix} \mathbb{I}_{3 \times 3} & -[\boldsymbol{\rho}]_{\times} \end{bmatrix} \begin{pmatrix} \delta \mathbf{r}_S \\ \delta \boldsymbol{\Phi} \end{pmatrix}$$

■ Applied to principle of virtual work

$$\begin{aligned} 0 = \delta W &= \int_B \begin{pmatrix} \delta \mathbf{r}_S \\ \delta \boldsymbol{\Phi} \end{pmatrix}^T \begin{bmatrix} \mathbb{I}_{3 \times 3} \\ [\boldsymbol{\rho}]_{\times} \end{bmatrix} \left(\begin{bmatrix} \mathbb{I}_{3 \times 3} & -[\boldsymbol{\rho}]_{\times} \end{bmatrix} \begin{pmatrix} \mathbf{a}_S \\ \dot{\boldsymbol{\Psi}} \end{pmatrix} dm + [\boldsymbol{\Omega}]_{\times}^2 \boldsymbol{\rho} dm - d\mathbf{F}_{ext} \right) \\ &= \begin{pmatrix} \delta \mathbf{r}_S \\ \delta \boldsymbol{\Phi} \end{pmatrix}^T \int_B \left(\begin{bmatrix} \mathbb{I}_{3 \times 3} dm & [\boldsymbol{\rho}]_{\times}^T dm \\ [\boldsymbol{\rho}]_{\times} dm & -[\boldsymbol{\rho}]_{\times}^2 dm \end{bmatrix} \begin{pmatrix} \mathbf{a}_S \\ \dot{\boldsymbol{\Psi}} \end{pmatrix} + \begin{pmatrix} [\boldsymbol{\Omega}]_{\times}^2 \boldsymbol{\rho} dm \\ [\boldsymbol{\rho}]_{\times} [\boldsymbol{\Omega}]_{\times}^2 \boldsymbol{\rho} dm \end{pmatrix} - \begin{pmatrix} d\mathbf{F}_{ext} \\ [\boldsymbol{\rho}]_{\times} d\mathbf{F}_{ext} \end{pmatrix} \right) \end{aligned}$$

$$0 = \delta W = \begin{pmatrix} \delta \mathbf{r}_S \\ \delta \boldsymbol{\Phi} \end{pmatrix}^T \left(\begin{bmatrix} \mathbb{I}_{3 \times 3} m & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Theta}_S \end{bmatrix} \begin{pmatrix} \mathbf{a}_S \\ \dot{\boldsymbol{\Psi}} \end{pmatrix} + \begin{pmatrix} \mathbf{0} \\ [\boldsymbol{\Omega}]_{\times} \boldsymbol{\Theta}_S \boldsymbol{\Omega} \end{pmatrix} - \begin{pmatrix} \mathbf{F}_{ext} \\ \mathbf{T}_{ext} \end{pmatrix} \right) \quad \forall \begin{pmatrix} \delta \mathbf{r}_S \\ \delta \boldsymbol{\Phi} \end{pmatrix}$$

$$\delta W = \int_B \delta \mathbf{r}^T \cdot (\ddot{\mathbf{r}} dm - d\mathbf{F}_{ext}) = 0$$



Impulse and angular momentum

- Use the following definitions

$$\mathbf{p}_S = m\mathbf{v}_S$$

linear momentum

$$\mathbf{N}_S = \mathbf{\Theta}_S \mathbf{\Omega}_S$$

angular momentum

$$\dot{\mathbf{p}}_S = m\mathbf{a}_S$$

change in linear momentum

$$\dot{\mathbf{N}}_S = \mathbf{\Theta}_S \mathbf{\Psi} + \mathbf{\Omega} \times \mathbf{\Theta}_S \mathbf{\Omega}$$

change in angular momentum

- Conservation of impulse and angular momentum

$$0 = \delta W = \begin{pmatrix} \delta \mathbf{r}_s \\ \delta \mathbf{\Phi} \end{pmatrix}^T \left(\begin{array}{c|c} \boxed{\dot{\mathbf{p}}_S} & \boxed{\mathbf{F}_{ext}} \\ \hline \boxed{\dot{\mathbf{N}}_S} & \boxed{\mathbf{T}_{ext}} \end{array} \right) \quad \forall \begin{pmatrix} \delta \mathbf{r}_s \\ \delta \mathbf{\Phi} \end{pmatrix}$$

Newton A free body can move
In all directions

Euler External forces and moments

Change in impulse and angular momentum

$$\begin{aligned} \dot{\mathbf{p}}_S &= \mathbf{F}_{ext} \\ \dot{\mathbf{N}}_S &= \mathbf{T}_{ext} \end{aligned}$$

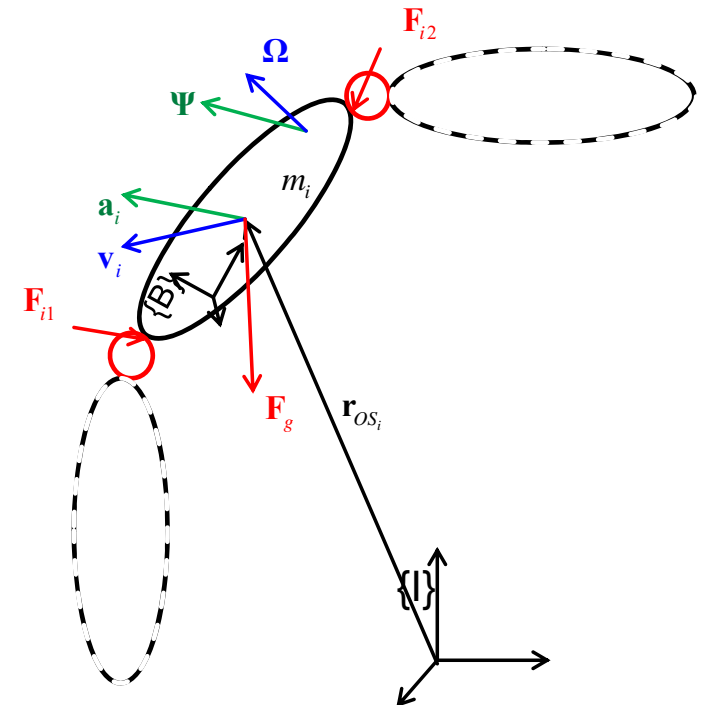
1st Method for EoM

Newton-Euler for single bodies

- Cut all bodies free
- Introduction of constraining force
- Apply conservation \mathbf{p} and \mathbf{N} to individual bodies

- System of equations
 - 6n equation
 - Eliminate all constrained forces (5n)

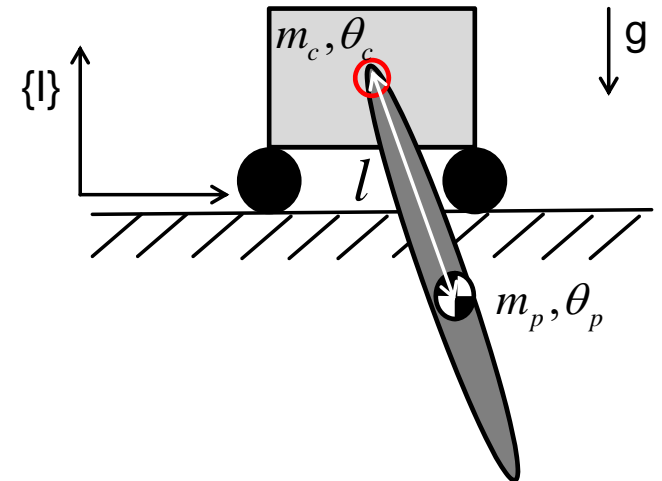
- Pros and Cons
 - + Intuitively clear
 - + Direct access to constraining forces
 - Becomes a huge combinatorial problem for large MBS



Free Cut

Cart pendulum example

- Find the equation of motion



Newton-Euler in Generalized Motion Directions

- For multi-body systems $0 = \delta W = \sum_{i=1}^{n_b} \begin{pmatrix} \delta \mathbf{r}_{S_i} \\ \delta \Phi_{S_i} \end{pmatrix}^T \left(\begin{pmatrix} \dot{\mathbf{p}}_{S_i} \\ \dot{\mathbf{N}}_{S_i} \end{pmatrix} - \begin{pmatrix} \mathbf{F}_{ext,i} \\ \mathbf{T}_{ext,i} \end{pmatrix} \right) \quad \forall \begin{pmatrix} \delta \mathbf{r}_s \\ \delta \Phi_{S_i} \end{pmatrix}_{\text{consistent}}$
- Express the impulse/angular momentum in generalized coordinates

$$\begin{pmatrix} \mathbf{v}_s \\ \boldsymbol{\Omega} \end{pmatrix} = \begin{bmatrix} \mathbf{J}_P \\ \mathbf{J}_R \end{bmatrix} \dot{\mathbf{q}} \quad \begin{pmatrix} \dot{\mathbf{p}}_{S_i} \\ \dot{\mathbf{N}}_{S_i} \end{pmatrix} = \begin{pmatrix} m \mathbf{a}_{S_i} \\ \boldsymbol{\Theta}_{S_i} \boldsymbol{\Psi}_{S_i} + \boldsymbol{\Omega}_{S_i} \times \boldsymbol{\Theta}_{S_i} \boldsymbol{\Omega}_{S_i} \end{pmatrix}$$

$$\begin{pmatrix} \mathbf{a}_s \\ \boldsymbol{\Psi} \end{pmatrix} = \begin{bmatrix} \mathbf{J}_P \\ \mathbf{J}_R \end{bmatrix} \ddot{\mathbf{q}} + \begin{bmatrix} \dot{\mathbf{J}}_P \\ \dot{\mathbf{J}}_R \end{bmatrix} \dot{\mathbf{q}} \quad = \begin{pmatrix} m \mathbf{J}_{S_i} \\ \boldsymbol{\Theta}_{S_i} \mathbf{J}_{R_i} \end{pmatrix} \ddot{\mathbf{q}} + \begin{pmatrix} m \dot{\mathbf{J}}_{S_i} \dot{\mathbf{q}} \\ \boldsymbol{\Theta}_{S_i} \dot{\mathbf{J}}_{R_i} \dot{\mathbf{q}} + \mathbf{J}_{R_i} \dot{\mathbf{q}} \times \boldsymbol{\Theta}_{S_i} \mathbf{J}_{R_i} \dot{\mathbf{q}} \end{pmatrix}$$
- Virtual displacement in generalized coordinates

$$\begin{pmatrix} \delta \mathbf{r}_s \\ \delta \Phi \end{pmatrix} = \begin{bmatrix} \mathbf{J}_P \\ \mathbf{J}_R \end{bmatrix} \delta \mathbf{q}$$
- With this, the principle of virtual work transforms to

$$0 = \delta W = \delta \mathbf{q}^T \sum_{i=1}^{n_b} \underbrace{\begin{pmatrix} \mathbf{J}_{S_i} \\ \mathbf{J}_{R_i} \end{pmatrix}^T \begin{pmatrix} m \mathbf{J}_{S_i} \\ \boldsymbol{\Theta}_{S_i} \mathbf{J}_{R_i} \end{pmatrix}}_{\mathbf{M}(\mathbf{q})} \ddot{\mathbf{q}} + \underbrace{\begin{pmatrix} \mathbf{J}_{S_i} \\ \mathbf{J}_{R_i} \end{pmatrix}^T \begin{pmatrix} m \dot{\mathbf{J}}_{S_i} \dot{\mathbf{q}} \\ \boldsymbol{\Theta}_{S_i} \dot{\mathbf{J}}_{R_i} \dot{\mathbf{q}} + \mathbf{J}_{R_i} \dot{\mathbf{q}} \times \boldsymbol{\Theta}_{S_i} \mathbf{J}_{R_i} \dot{\mathbf{q}} \end{pmatrix}}_{\mathbf{b}(\mathbf{q}, \dot{\mathbf{q}})} - \underbrace{\begin{pmatrix} \mathbf{J}_{P_i} \\ \mathbf{J}_{R_i} \end{pmatrix}^T \begin{pmatrix} \mathbf{F}_{ext,i} \\ \mathbf{T}_{ext,i} \end{pmatrix}}_{\mathbf{g}(\mathbf{q})} \quad \forall \delta \mathbf{q}$$

Projected Newton-Euler

- Equation of motion $\mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{b}(\mathbf{q}, \dot{\mathbf{q}}) + \mathbf{g}(\mathbf{q}) = \mathbf{0}$
- Directly get the dynamic properties of a multi-body system with n bodies

$$\mathbf{M} = \sum_{i=1}^{n_b} \left({}^{\mathcal{A}}\mathbf{J}_{S_i}^T \cdot m \cdot {}^{\mathcal{A}}\mathbf{J}_{S_i} + {}^{\mathcal{B}}\mathbf{J}_{R_i}^T \cdot {}^{\mathcal{B}}\boldsymbol{\Theta}_{S_i} \cdot {}^{\mathcal{B}}\mathbf{J}_{R_i} \right)$$

$$\mathbf{b} = \sum_{i=1}^{n_b} \left({}^{\mathcal{A}}\mathbf{J}_{S_i}^T \cdot m \cdot {}^{\mathcal{A}}\dot{\mathbf{J}}_{S_i} \cdot \dot{\mathbf{q}} + {}^{\mathcal{B}}\mathbf{J}_{R_i}^T \cdot \left({}^{\mathcal{B}}\boldsymbol{\Theta}_{S_i} \cdot {}^{\mathcal{B}}\dot{\mathbf{J}}_{R_i} \cdot \dot{\mathbf{q}} + {}^{\mathcal{B}}\boldsymbol{\Omega}_{S_i} \times {}^{\mathcal{B}}\boldsymbol{\Theta}_{S_i} \cdot {}^{\mathcal{B}}\boldsymbol{\Omega}_{S_i} \right) \right)$$

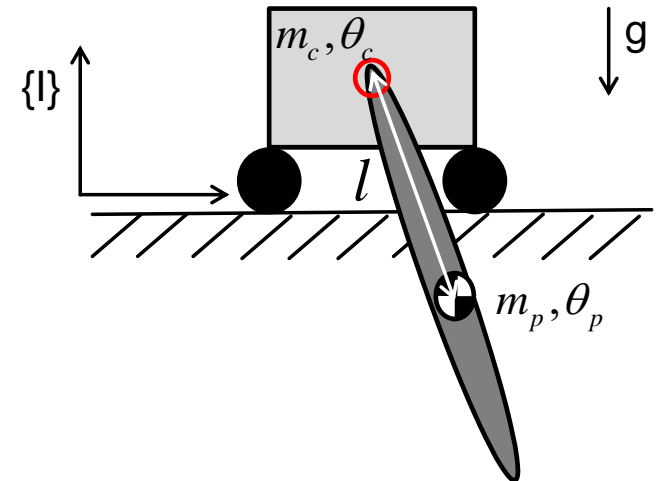
$$\mathbf{g} = \sum_{i=1}^{n_b} \left(-{}^{\mathcal{A}}\mathbf{J}_{S_i}^T {}^{\mathcal{A}}\mathbf{F}_{g,i} \right)$$

- For actuated systems, include actuation force as external force for each body
 - If actuators act in the direction of generalized coordinates, $\boldsymbol{\tau}$ corresponds to stacked actuator commands

Projected Newton-Euler

Cart pendulum example

- Find the equation of motion



3rd Method for EoM

Lagrange II

- Lagrangian $\mathcal{L} = \mathcal{T} - \mathcal{U}$

↙ kinetic energy

↘ potential energy

- Lagrangian equation $\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}} \right) - \left(\frac{\partial \mathcal{L}}{\partial \mathbf{q}} \right) = \boldsymbol{\tau}$

- Since $\mathcal{U} = \mathcal{U}(\mathbf{q})$
 $\mathcal{T} = \mathcal{T}(\mathbf{q}, \dot{\mathbf{q}})$

inertial forces

gravity vector

$$\boxed{\frac{d}{dt} \left(\frac{\partial \mathcal{T}}{\partial \dot{\mathbf{q}}} \right) - \frac{\partial \mathcal{T}}{\partial \mathbf{q}}} + \boxed{\frac{\partial \mathcal{U}}{\partial \mathbf{q}}} = \boldsymbol{\tau}$$

$$\mathcal{T} = \frac{1}{2} \dot{\mathbf{q}}^T \mathbf{M} \dot{\mathbf{q}}$$

with $\frac{\partial \mathcal{T}}{\partial \dot{\mathbf{q}}} = \mathbf{M} \dot{\mathbf{q}}$

$$\mathbf{M} \ddot{\mathbf{q}} + \mathbf{M} \dot{\mathbf{q}} - \frac{1}{2} \begin{pmatrix} \dot{\mathbf{q}}^T \frac{\partial \mathbf{M}}{\partial q_1} \dot{\mathbf{q}} \\ \vdots \\ \dot{\mathbf{q}}^T \frac{\partial \mathbf{M}}{\partial q_n} \dot{\mathbf{q}} \end{pmatrix} + \vec{g} = \mathbf{M}(\mathbf{q}) \ddot{\mathbf{q}} + \vec{b}(\mathbf{q}, \dot{\mathbf{q}}) + \vec{g}(\mathbf{q}) = \boldsymbol{\tau}$$

Lagrange II

Kinetic energy

- Kinetic energy in joint space

$$\mathcal{T} = \frac{1}{2} \dot{\mathbf{q}}^T \mathbf{M} \dot{\mathbf{q}}$$

- Kinetic energy for all bodies

$$\mathcal{T} = \sum_{i=1}^{n_b} \left(\frac{1}{2} m_i \mathbf{A} \dot{\mathbf{r}}_{S_i}^T \mathbf{A} \dot{\mathbf{r}}_{S_i} + \frac{1}{2} \mathbf{B} \boldsymbol{\Omega}_{S_i}^T \cdot \mathbf{B} \boldsymbol{\Theta}_{S_i} \cdot \mathbf{B} \boldsymbol{\Omega}_{S_i} \right)$$

- From kinematics we know that

$$\begin{aligned} \dot{\mathbf{r}}_{S_i} &= \mathbf{J}_{S_i} \dot{\mathbf{q}} \\ \boldsymbol{\Omega}_{S_i} &= \mathbf{J}_{R_i} \dot{\mathbf{q}} \end{aligned}$$

- Hence we get

$$\mathcal{T}(\mathbf{q}, \dot{\mathbf{q}}) = \frac{1}{2} \dot{\mathbf{q}}^T \underbrace{\left(\sum_{i=1}^{n_b} (\mathbf{J}_{S_i}^T m \mathbf{J}_{S_i} + \mathbf{J}_{R_i}^T \boldsymbol{\Theta}_{S_i} \mathbf{J}_{R_i}) \right)}_{\mathbf{M}(\mathbf{q})} \dot{\mathbf{q}}$$

Lagrange II

Potential energy

- Two sources for potential forces

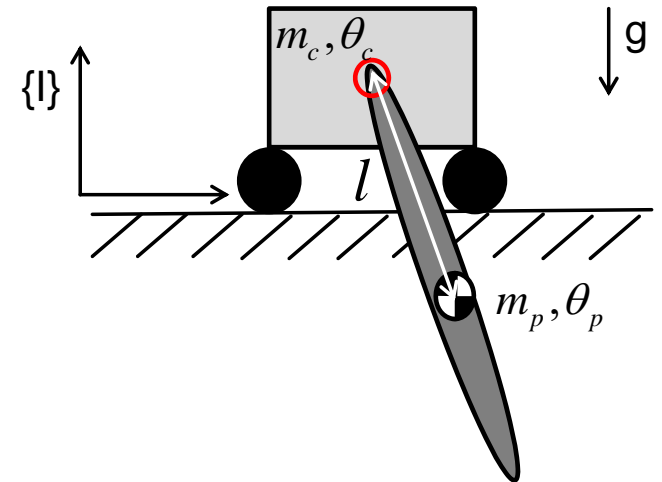
- Gravitational forces $\mathbf{F}_{g_i} = m_i g \mathbf{e}_g \longrightarrow \mathcal{U}_g = - \sum_{i=1}^{n_b} \mathbf{r}_{S_i}^T \mathbf{F}_{g_i}$

- Spring forces $\mathbf{F}_E = k_j (\|\mathbf{r} - \mathbf{r}_0\| - d_0) \frac{\mathbf{r} - \mathbf{r}_0}{\|\mathbf{r} - \mathbf{r}_0\|} \longrightarrow \mathcal{U}_{E_j} = \frac{1}{2} k_j (d(\mathbf{q}) - d_0)^2$

Lagrange II

Cart pendulum example

- Find the equation of motion



External Forces

- Given: $n_{f,ext}$ external forces \mathbf{F}_j

- Generalized forces are calculated as:
$$\boldsymbol{\tau}_{F,ext} = \sum_{j=1}^{n_{f,ext}} \mathbf{J}_{P,j}^T \mathbf{F}_j$$

- Given: $n_{m,ext}$ external torques \mathbf{T}_k

- Generalized forces are calculated
$$\boldsymbol{\tau}_{T,ext} = \sum_{k=1}^{n_{m,ext}} \mathbf{J}_{R,k}^T \mathbf{T}_{ext,k}$$

- For actuator torques:
$$\boldsymbol{\tau}_{a,k} = (\mathbf{J}_{S_k} - \mathbf{J}_{S_{k-1}})^T \mathbf{F}_{a,k} + (\mathbf{J}_{R_k} - \mathbf{J}_{R_{k-1}})^T \mathbf{T}_{a,k}$$

External Forces

Cart pendulum example

- Equation of motion without actuation

$$\underbrace{\begin{bmatrix} m_c + m_p & lm_p \cos(\varphi) \\ lm_p \cos(\varphi) & m_p l^2 + \theta_p \end{bmatrix}}_{\mathbf{M}} \ddot{\mathbf{q}} + \underbrace{\begin{pmatrix} -\dot{\varphi}^2 lm_p \sin(\varphi) \\ 0 \end{pmatrix}}_{\mathbf{b}} + \underbrace{\begin{pmatrix} 0 \\ m_p gl \sin(\varphi) \end{pmatrix}}_{\mathbf{g}} = \mathbf{0}$$

- Add actuator for the pendulum

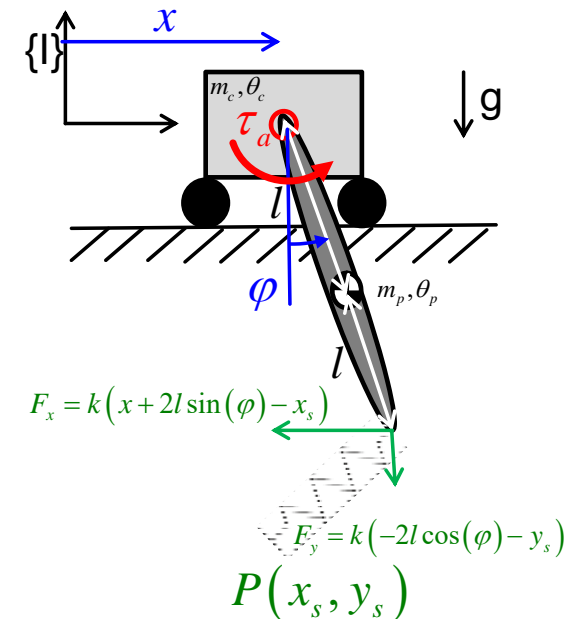
- Action on pendulum $T_p = \tau_a$
- Reaction on cart $T_c = -\tau_a$

$$\left. \begin{array}{l} \mathbf{J}_{Rp} = \begin{bmatrix} 0 & 1 \end{bmatrix} \\ \mathbf{J}_{Rc} = \begin{bmatrix} 0 & 0 \end{bmatrix} \end{array} \right\} \boldsymbol{\tau} = \sum \mathbf{J}_{R,i}^T \mathbf{T}_i = \mathbf{J}_{Rc}^T \mathbf{T}_c + \mathbf{J}_{Rp}^T \mathbf{T}_p = \begin{pmatrix} 0 \\ \tau_a \end{pmatrix}$$

- Add spring to the pendulum

- (world attachment point P, zero length 0, stiffness k)
- Action on pendulum

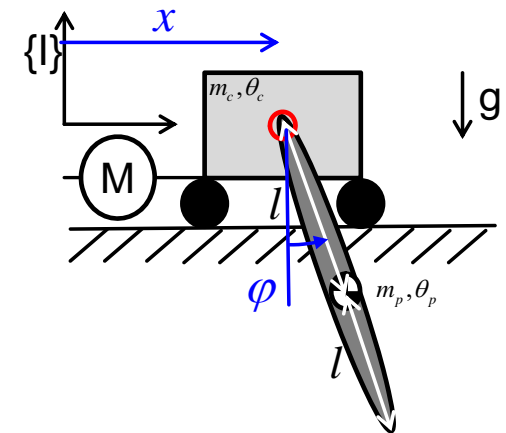
$$\left. \begin{array}{l} \mathbf{F}_s = \begin{pmatrix} -F_x \\ -F_y \end{pmatrix} \\ \mathbf{r} = \begin{pmatrix} x + 2l \sin(\varphi) \\ -2l \cos(\varphi) \end{pmatrix} \\ \mathbf{J}_s = \frac{\partial \mathbf{r}_s}{\partial \mathbf{q}} = \begin{bmatrix} 1 & 2l \cos(\varphi) \\ 0 & 2l \sin(\varphi) \end{bmatrix} \end{array} \right\} \boldsymbol{\tau} = \mathbf{J}^T \mathbf{F}_s = \begin{pmatrix} -F_x \\ -2l(F_x \cos(\varphi) + F_y \sin(\varphi)) \end{pmatrix}$$



External Forces

Cart pendulum example

- What is the external force coming from the motor



External Forces

Cart pendulum example

- What is the external force coming from the motor

- Action on cart F_{act}

$$F_c = F_{act} \quad \mathbf{J}_{pc} = \begin{bmatrix} 1 & 0 \end{bmatrix} \quad \boldsymbol{\tau} = \mathbf{J}^T F_c = \begin{pmatrix} F_{act} \\ 0 \end{pmatrix}$$

