ETH zürich

Lecture «Robot Dynamics»: Kinematics 1

151-0851-00 V

lecture:	CAB G11	Tuesday 10:15 – 12:00, every week
exercise:	HG E1.2	Wednesday 8:15 – 10:00, according to schedule (about every 2nd week)

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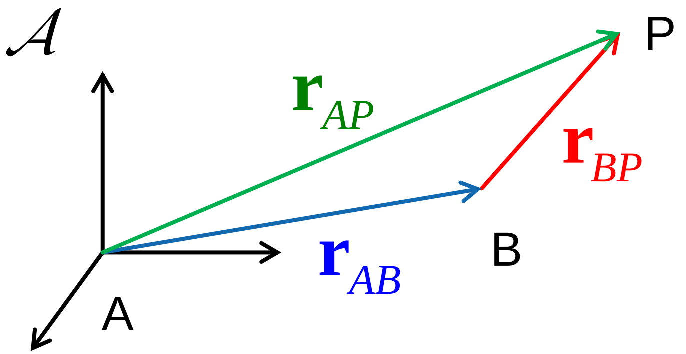
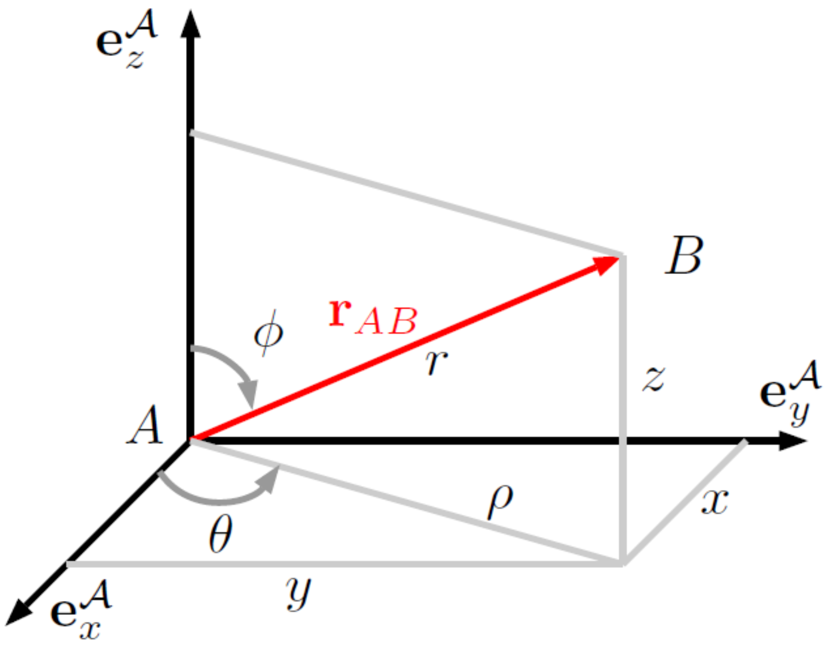
Recapitulation: Vectors, Position, and Vector Calculus

- Builds upon notation of other dynamics classes at ETH and IEEE standards

- Vector: $\mathbf{r}$ (often also $\vec{r}$)
- Vector from point B to P: $\mathbf{r}_{BP}$
- Reference coordinate system $\mathcal{A}$ (calligraphic)
  $\left(\mathrm{e}_x^{\mathcal{A}}, \mathrm{e}_y^{\mathcal{A}}, \mathrm{e}_z^{\mathcal{A}}\right)$:= orthonormal basis of R^3
- Numerical representation of a vector: ${}_{\mathcal{A}}\mathbf{r}_{BP}$
- Addition of vectors: $\mathbf{r}_{AP} = \mathbf{r}_{AB} + \mathbf{r}_{BP}$
- Use the same reference frame: ${}_{\mathcal{A}}\mathbf{r}_{AP} = {}_{\mathcal{A}}\mathbf{r}_{AB} + {}_{\mathcal{A}}\mathbf{r}_{BP}$

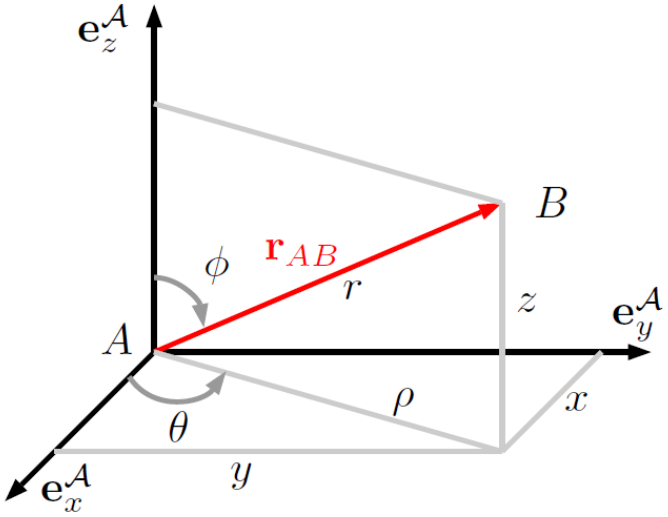


Parameterization of Vectors



- Cartesian coordinates $\chi_{Pc} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$
  - Position vector ${}_{\mathcal{A}}\mathbf{r} = x\mathbf{e}_x^{\mathcal{A}} + y\mathbf{e}_y^{\mathcal{A}} + z\mathbf{e}_z^{\mathcal{A}} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$
- Cylindrical coordinates $\chi_{Pz} = \begin{pmatrix} \rho \\ \theta \\ z \end{pmatrix}$
  - Position vector ${}_{\mathcal{A}}\mathbf{r} = \begin{pmatrix} \rho\cos\theta \\ \rho\sin\theta \\ z \end{pmatrix}$
- Spherical coordinates $\chi_{Ps} = \begin{pmatrix} r \\ \theta \\ \phi \end{pmatrix}$
  - Position vector ${}_{\mathcal{A}}\mathbf{r} = \begin{pmatrix} r\cos\theta\sin\phi \\ r\sin\theta\sin\phi \\ r\cos\phi \end{pmatrix}$

Parameterization of Vectors
Example



$$_{\mathcal{A}}\mathbf{r}_{AP} = {}_{\mathcal{A}}\mathbf{r}_{AB} + {}_{\mathcal{A}}\mathbf{r}_{BP}$$

$$\begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}1\\0\\0\end{pmatrix} + \begin{pmatrix}0\\1\\1\end{pmatrix}$$

$$\begin{pmatrix}1\\0\\0\end{pmatrix}: \begin{cases}\chi_{Pc} = (1,0,0)^T\\ \chi_{Pz} = (1,0,0)^T\\ \chi_{Ps} = \left(1,0,\frac{\pi}{2}\right)^T\end{cases} \quad \begin{pmatrix}0\\1\\1\end{pmatrix}: \begin{cases}\chi_{Pc} = (0,1,1)^T\\ \chi_{Pz} = \left(1,\frac{\pi}{2},1\right)^T\\ \chi_{Ps} = \left(\sqrt{2},\frac{\pi}{2},\frac{\pi}{4}\right)^T\end{cases} \quad \begin{pmatrix}1\\1\\1\end{pmatrix}: \begin{cases}\chi_{Pc} = (1,1,1)^T\\ \chi_{Pz} = \left(\sqrt{2},\frac{\pi}{4},1\right)^T\\ \chi_{Ps} = \left(\sqrt{3},\frac{\pi}{4},\arccos\left(\frac{1}{\sqrt{3}}\right)\right)^T\end{cases}$$

- Only for Cartesian coordinates it holds that $\chi_{AP} = \chi_{AB} + \chi_{BP}$
- NEVER do this for other representations (requires special algebra!!)
  => *we will encounter similar problems for rotations*

Differentiation of Representation ⇔ Linear Velocity

- The velocity of point P relative to point B, expressed in frame $\mathcal{A}$ is: $_\mathcal{A}\dot{\mathbf{r}}_{BP}$

- Question: What is the relationship between the velocity $\dot{\mathbf{r}}$ and the time derivative of the representation $\dot{\chi}$

$$\mathbf{r} = \mathbf{r}(\chi)$$

$$\dot{\mathbf{r}} = \frac{\partial \mathbf{r}}{\partial \chi}\dot{\chi}$$

$$\Rightarrow$$

$$\dot{\mathbf{r}} = \mathbf{E}_P(\chi)\cdot\dot{\chi}$$

$$\dot{\chi} = \mathbf{E}_P^{-1}(\chi)\cdot\dot{\mathbf{r}}$$

Differentiation of Representation ⇔ Linear Velocity

- Cartesian coordinates: $\mathbf{E}_{Pc}(\chi_{Pc}) = \mathbf{E}_{Pc}^{-1}(\chi_{Pc}) = \mathbb{I}$

- Cylindrical coordinates: $_{\mathcal{A}}\mathbf{r} = \begin{pmatrix} \rho\cos\theta \\ \rho\sin\theta \\ z \end{pmatrix}$ $\quad \dot{\mathbf{r}}(\chi_{Pz}) = \begin{pmatrix} \dot\rho\cos\theta - \rho\dot\theta\sin\theta \\ \dot\rho\sin\theta + \rho\dot\theta\cos\theta \\ \dot z \end{pmatrix}$

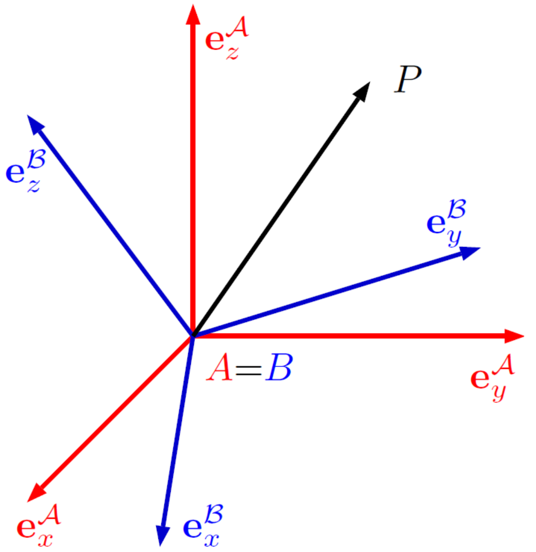
$$\dot{\chi}_{Pz} = \begin{pmatrix} \dot\rho \\ \dot\theta \\ \dot z \end{pmatrix} = \begin{pmatrix} \dot x\cos\theta + \dot y\sin\theta \\ -\dot x\sin\theta/\rho + \dot y\cos\theta/\rho \\ \dot z \end{pmatrix} = \underbrace{\begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta/\rho & \cos\theta/\rho & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\mathbf{E}_{Pz}^{-1}} \begin{pmatrix} \dot x \\ \dot y \\ \dot z \end{pmatrix}$$

$$\mathbf{E}_{Pz}(\chi_{Pz}) = \frac{\partial\mathbf{r}(\chi_{Pz})}{\partial\chi_{Pz}} = \begin{bmatrix} \cos\theta & -\rho\sin\theta & 0 \\ \sin\theta & \rho\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Rotations

- Position of P with respect to A expressed in $\mathcal{A}$: $${}_{\mathcal{A}}\mathbf{r}_{AP} = \begin{pmatrix} {}_{\mathcal{A}}r_{AP_x} \\ {}_{\mathcal{A}}r_{AP_y} \\ {}_{\mathcal{A}}r_{AP_z} \end{pmatrix}$$
- Position of P with respect to A expressed in $\mathcal{B}$: $${}_{\mathcal{B}}\mathbf{r}_{AP} = \begin{pmatrix} {}_{\mathcal{B}}r_{AP_x} \\ {}_{\mathcal{B}}r_{AP_y} \\ {}_{\mathcal{B}}r_{AP_z} \end{pmatrix}$$
- Write the unit vectors of $\mathcal{B}$ expressed in $\mathcal{A}$ as matrix: $\left[{}_{\mathcal{A}}\mathbf{e}_x^{\mathcal{B}}, {}_{\mathcal{A}}\mathbf{e}_y^{\mathcal{B}}, {}_{\mathcal{A}}\mathbf{e}_z^{\mathcal{B}}\right]$
- =>

$$
\begin{aligned}
{}_{\mathcal{A}}\mathbf{r}_{AP} &= {}_{\mathcal{A}}\mathbf{e}_x^{\mathcal{B}} \cdot {}_{\mathcal{B}}r_{AP_x} + {}_{\mathcal{A}}\mathbf{e}_y^{\mathcal{B}} \cdot {}_{\mathcal{B}}r_{AP_y} + {}_{\mathcal{A}}\mathbf{e}_z^{\mathcal{B}} \cdot {}_{\mathcal{B}}r_{AP_z} \\
{}_{\mathcal{A}}\mathbf{r}_{AP} &= \left[{}_{\mathcal{A}}\mathbf{e}_x^{\mathcal{B}} \quad {}_{\mathcal{A}}\mathbf{e}_y^{\mathcal{B}} \quad {}_{\mathcal{A}}\mathbf{e}_z^{\mathcal{B}}\right] \cdot {}_{\mathcal{B}}\mathbf{r}_{AP} \\
&= \mathbf{C}_{\mathcal{AB}} \cdot {}_{\mathcal{B}}\mathbf{r}_{AP}.
\end{aligned}
$$



Rotation Matrix

- The rotation matrix transforms vectors expressed in $\mathcal{B}$ to $\mathcal{A}$:

$$\mathbf{C}_{\mathcal{AB}} = \left[{}_{\mathcal{A}}\mathbf{e}_x^{\mathcal{B}} \quad {}_{\mathcal{A}}\mathbf{e}_y^{\mathcal{B}} \quad {}_{\mathcal{A}}\mathbf{e}_z^{\mathcal{B}} \right] \qquad {}_{\mathcal{A}}\mathbf{u} = \mathbf{C}_{\mathcal{AB}} \cdot {}_{\mathcal{B}}\mathbf{u}$$

- The matrix is orthogonal: $\mathbf{C}_{\mathcal{BA}} = \mathbf{C}_{\mathcal{AB}}^{-1} = \mathbf{C}_{\mathcal{AB}}^{T}$

- Belongs to special orthonormal group SO(3) (and not R^3)
  - This causes difficulties and requires special algebra

- Consecutive rotations:

$$\begin{aligned} {}_{\mathcal{A}}\mathbf{u} &= \mathbf{C}_{\mathcal{AB}} \cdot {}_{\mathcal{B}}\mathbf{u}. \\ {}_{\mathcal{B}}\mathbf{u} &= \mathbf{C}_{\mathcal{BC}} \cdot {}_{\mathcal{C}}\mathbf{u}. \end{aligned} \quad \Longrightarrow \quad \begin{aligned} {}_{\mathcal{A}}\mathbf{u} &= \mathbf{C}_{\mathcal{AB}} \cdot (\mathbf{C}_{\mathcal{BC}} \cdot {}_{\mathcal{C}}\mathbf{u}) \\ &= \mathbf{C}_{\mathcal{AC}} \cdot {}_{\mathcal{C}}\mathbf{u}. \end{aligned} \qquad \boxed{\mathbf{C}_{\mathcal{AC}} = \mathbf{C}_{\mathcal{AB}} \cdot \mathbf{C}_{\mathcal{BC}}}$$

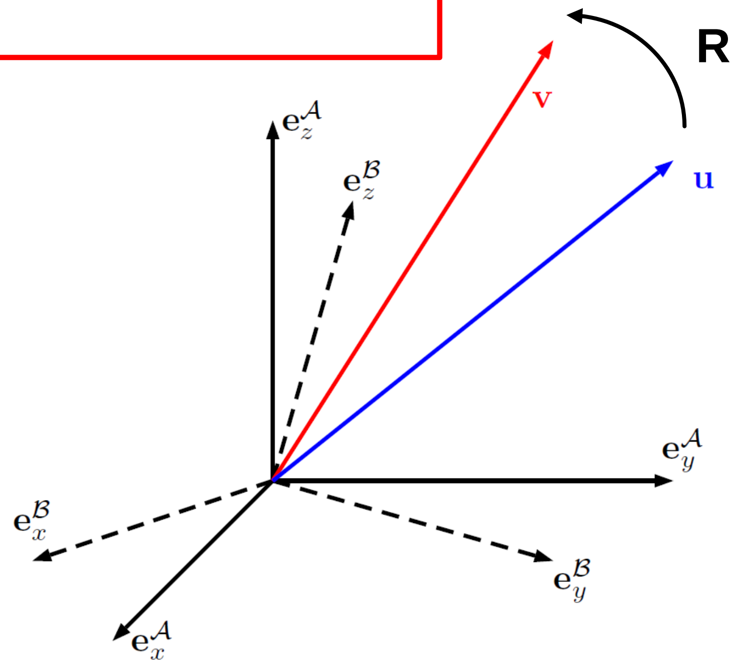
Passive and Active Rotation

- Passive rotation = mapping of the same vector from frame $\mathcal{B}$ to $\mathcal{A}$

$$_{\mathcal{A}}\mathbf{u} = \mathbf{C}_{\mathcal{A}\mathcal{B}} \cdot {_{\mathcal{B}}\mathbf{u}}$$

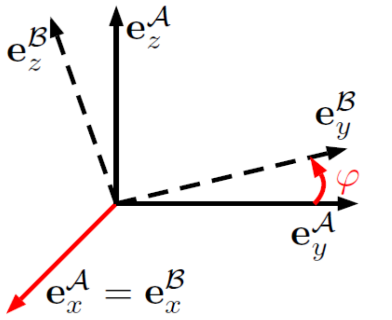
- Active rotation = rotating a vector in the same frame

$$_{\mathcal{A}}\mathbf{v} = \mathbf{R} \cdot {_{\mathcal{A}}\mathbf{u}}$$

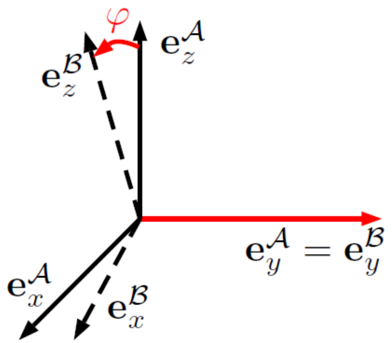


Elementary Rotation

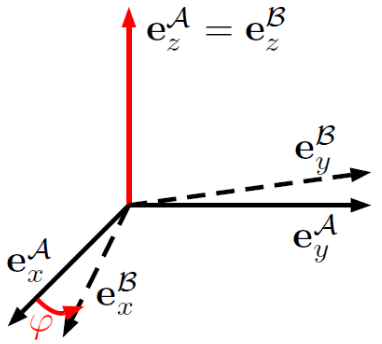
- Find the elementary rotation matrix s.t $\ _{\mathcal{A}}\mathbf{u} = \mathbf{C}_{\mathcal{AB}} \cdot \ _{\mathcal{B}}\mathbf{u}$



$$\mathbf{C}_{\mathcal{AB}} = \mathbf{C}_x(\varphi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{bmatrix}$$



$$\mathbf{C}_{\mathcal{AB}} = \mathbf{C}_y(\varphi) = \begin{bmatrix} \cos\varphi & 0 & \sin\varphi \\ 0 & 1 & 0 \\ -\sin\varphi & 0 & \cos\varphi \end{bmatrix}$$



$$\mathbf{C}_{\mathcal{AB}} = \mathbf{C}_z(\varphi) = \begin{bmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Homogeneous Transformation
Combined Translation and Rotation

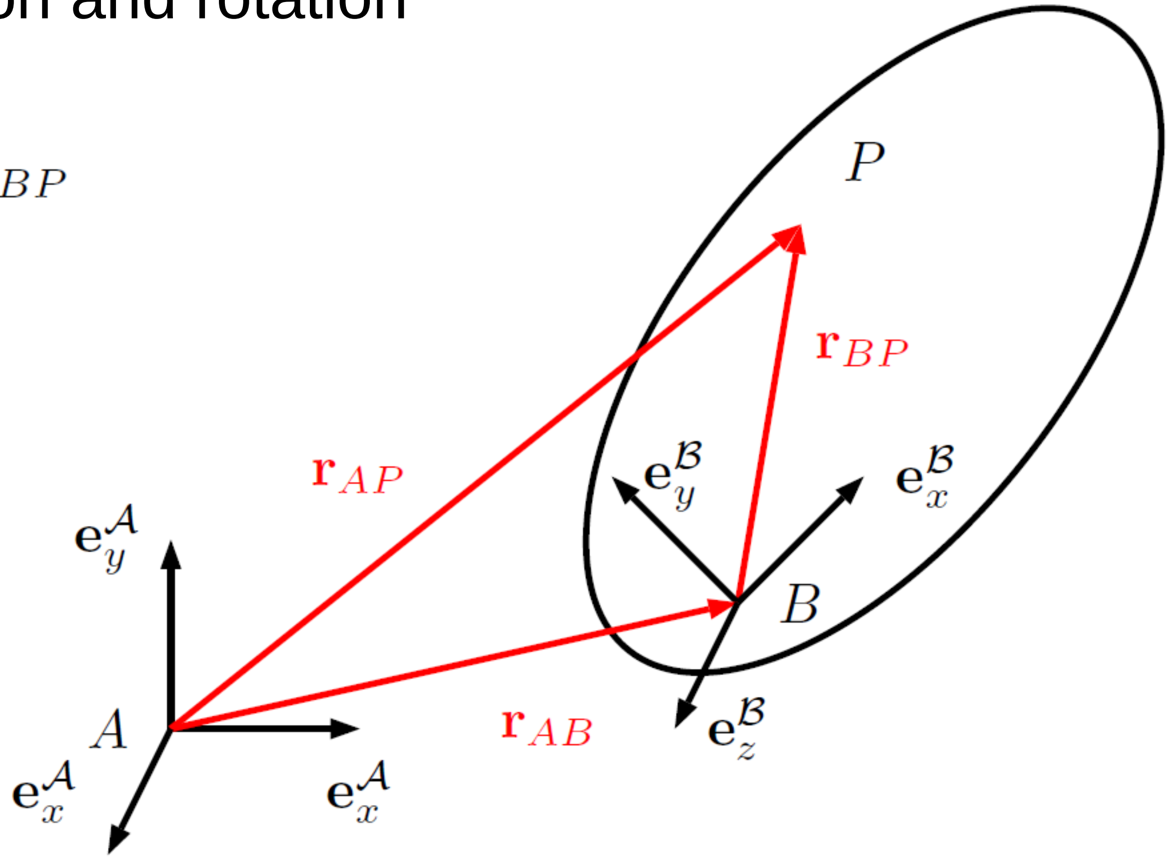
- Homogeneous transformation = translation and rotation

$$\mathbf{r}_{AP} = \mathbf{r}_{AB} + \mathbf{r}_{BP}$$

$$_{\mathcal{A}}\mathbf{r}_{AP} = {_{\mathcal{A}}\mathbf{r}_{AB}} + {_{\mathcal{A}}\mathbf{r}_{BP}} = {_{\mathcal{A}}\mathbf{r}_{AB}} + \mathbf{C}_{\mathcal{AB}} \cdot {_{\mathcal{B}}\mathbf{r}_{BP}}$$

$$\begin{pmatrix} _{\mathcal{A}}\mathbf{r}_{AP} \\ 1 \end{pmatrix} = \underbrace{\begin{bmatrix} \mathbf{C}_{\mathcal{AB}} & _{\mathcal{A}}\mathbf{r}_{AB} \\ \mathbf{0}_{1\times 3} & 1 \end{bmatrix}}_{\mathbf{T}_{\mathcal{AB}}} \begin{pmatrix} _{\mathcal{B}}\mathbf{r}_{BP} \\ 1 \end{pmatrix}$$

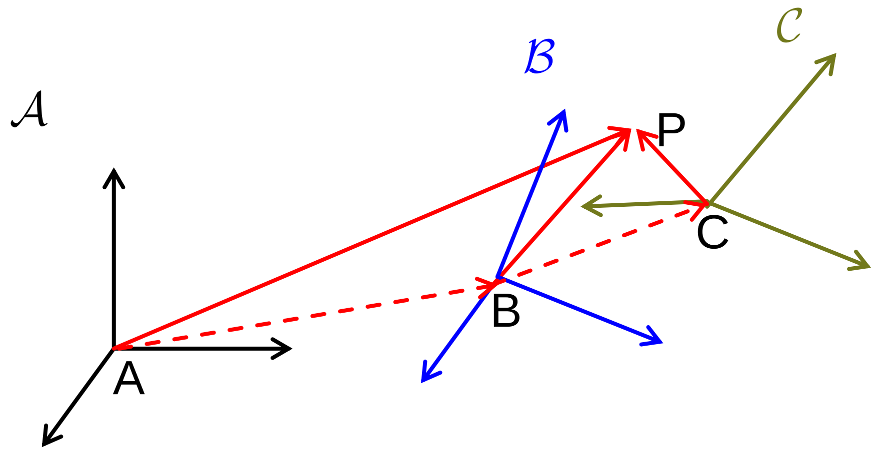
  - Inverse $\quad \mathbf{T}_{\mathcal{AB}}^{-1} = \begin{bmatrix} \mathbf{C}_{\mathcal{AB}}^T & \overbrace{-\mathbf{C}_{\mathcal{AB}}^T {_{\mathcal{A}}\mathbf{r}_{AB}}}^{_{\mathcal{B}}\mathbf{r}_{BA}} \\ \mathbf{0}_{1\times 3} & 1 \end{bmatrix}$



Homogeneous Transformations

Consecutive Transformation

$$\left.\begin{aligned} {}_{\mathcal{A}}\vec{r}_{AP} &= \mathbf{T}_{\mathcal{AB}} \cdot {}_{\mathcal{B}}\vec{r}_{BP} \\ {}_{\mathcal{B}}\vec{r}_{BP} &= \mathbf{T}_{\mathcal{BC}} \cdot {}_{\mathcal{C}}\vec{r}_{CP} \end{aligned}\right\} \mathbf{T}_{\mathcal{AC}} = \mathbf{T}_{\mathcal{AB}} \cdot \mathbf{T}_{\mathcal{BC}}$$



- This allows to transform an arbitrary vector between different reference frames (classical example: mapping of features in camera frame to world frame)

Homogeneous Transformation

Simple Example

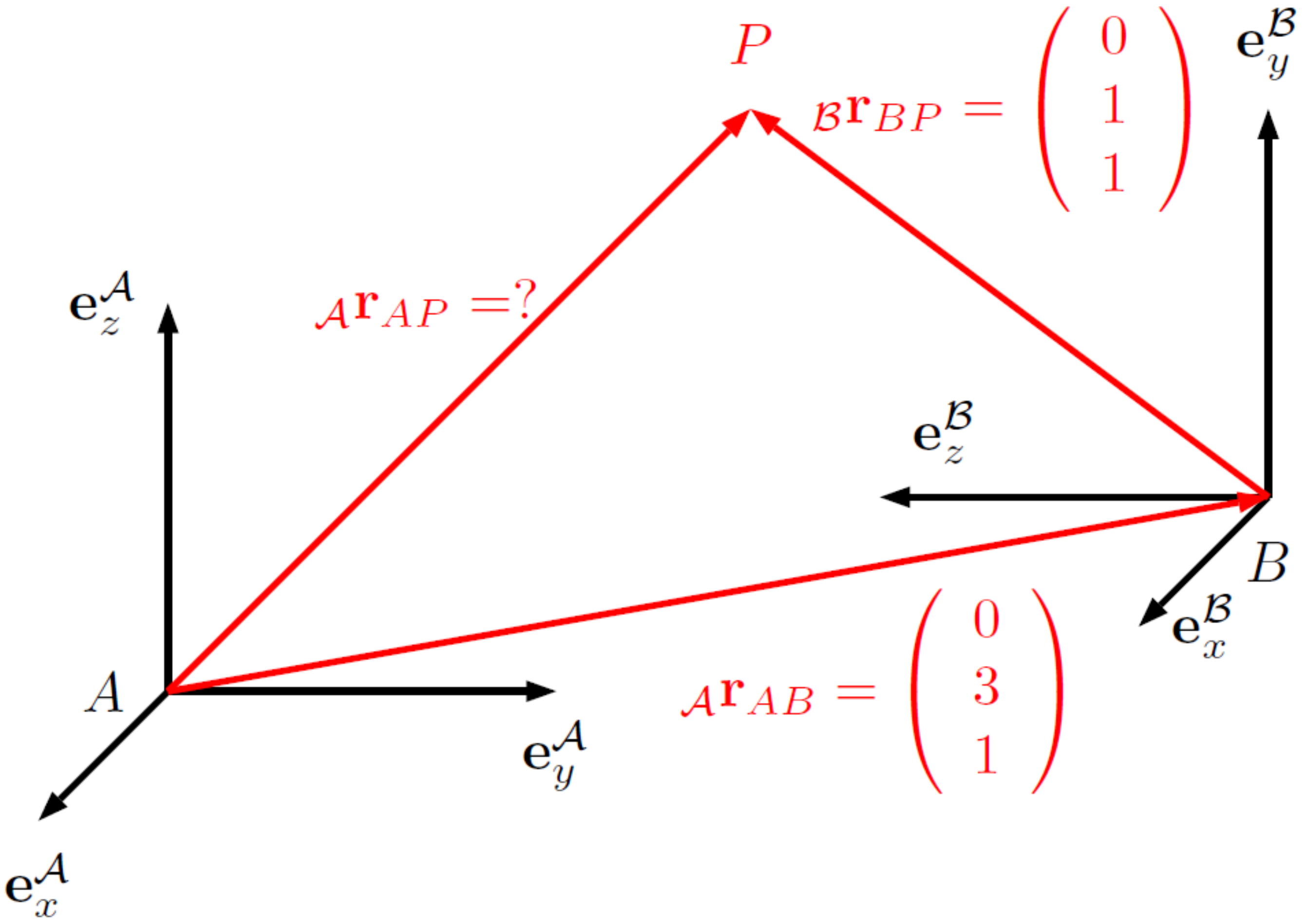
- Find the position vector ${}_{\mathcal{A}}\mathbf{r}_{AP}$
  - Find the transformation matrix

$$\mathbf{T}_{\mathcal{AB}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

  - Find the vector

$$\begin{pmatrix} {}_{\mathcal{A}}\vec{r}_{AP} \\ 1 \end{pmatrix} = \mathbf{T}_{\mathcal{AB}} \begin{pmatrix} {}_{\mathcal{B}}\vec{r}_{BP} \\ 1 \end{pmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 2 \\ 1 \end{pmatrix}$$



Angular Velocity

- Angular velocity ${}_{\mathcal{A}}\boldsymbol{\omega}_{\mathcal{A}\mathcal{B}}$ describes the relative rotational velocity of $\mathcal{B}$ wrt. $\mathcal{A}$ expressed in frame $\mathcal{A}$
- The relative velocity of $\mathcal{A}$ wrt. $\mathcal{B}$ is: $\boldsymbol{\omega}_{\mathcal{B}\mathcal{A}} = -\boldsymbol{\omega}_{\mathcal{A}\mathcal{B}}$
- Given the rotation matrix $\mathbf{C}_{\mathcal{A}\mathcal{B}}(t)$ between two frames, the angular velocity is

$$[{}_{\mathcal{A}}\boldsymbol{\omega}_{\mathcal{A}\mathcal{B}}]_{\times} = \dot{\mathbf{C}}_{\mathcal{A}\mathcal{B}} \cdot \mathbf{C}_{\mathcal{A}\mathcal{B}}^T \qquad [{}_{\mathcal{A}}\boldsymbol{\omega}_{\mathcal{A}\mathcal{B}}]_{\times} = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}, \qquad {}_{\mathcal{A}}\boldsymbol{\omega}_{\mathcal{A}\mathcal{B}} = \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$$

- Transformation of angular velocity: ${}_{\mathcal{B}}\boldsymbol{\omega}_{\mathcal{A}\mathcal{B}} = \mathbf{C}_{\mathcal{B}\mathcal{A}} \cdot {}_{\mathcal{A}}\boldsymbol{\omega}_{\mathcal{A}\mathcal{B}}$
- Addition of relative velocities: ${}_{\mathcal{D}}\boldsymbol{\omega}_{\mathcal{A}\mathcal{C}} = {}_{\mathcal{D}}\boldsymbol{\omega}_{\mathcal{A}\mathcal{B}} + {}_{\mathcal{D}}\boldsymbol{\omega}_{\mathcal{B}\mathcal{C}}$

Angular Velocity
Simple Example

- Given the rotation matrix $\mathbf{C}_{\mathcal{AB}}(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\alpha(t)) & \sin(\alpha(t)) \\ 0 & -\sin(\alpha(t)) & \cos(\alpha(t)) \end{bmatrix}$

$$\begin{aligned} [{}_{\mathcal{A}}\omega_{\mathcal{AB}}]_\times &= \dot{\mathbf{C}}_{\mathcal{AB}}\mathbf{C}_{\mathcal{AB}}^T \\ &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\dot{\alpha}\sin\alpha & \dot{\alpha}\cos\alpha \\ 0 & -\dot{\alpha}\cos\alpha & -\dot{\alpha}\sin\alpha \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \dot{\alpha} \\ 0 & -\dot{\alpha} & 0 \end{bmatrix} \end{aligned}$$

=> $${}_{\mathcal{A}}\omega_{\mathcal{AB}} = \begin{pmatrix} \dot{\alpha} \\ 0 \\ 0 \end{pmatrix}$$

Outlook (next week)
Rotation Parameterization

- Rotation matrix:
  - 3x3 = 9 parameters
  - Orthogonality = 6 constraints

$$\mathbf{C}_{\mathcal{AB}} = \begin{bmatrix} {}_{\mathcal{A}}\mathbf{e}_x^{\mathcal{B}} & {}_{\mathcal{A}}\mathbf{e}_y^{\mathcal{B}} & {}_{\mathcal{A}}\mathbf{e}_z^{\mathcal{B}} \end{bmatrix}$$

- Euler Angles
  - 3 parameters
  - singularity problem

$$\chi_{R,eulerZYX} = \begin{pmatrix} z \\ y \\ x \end{pmatrix}$$

- Angle Axis
  - 4 parameters (angle and axis)
  - unitary constraint

$$\chi_{R,AngleAxis} = \begin{pmatrix} \theta \\ \mathbf{n} \end{pmatrix}$$

- Quaternions
  - 4 parameters
  - no singularity constraints

$$\chi_{R,quat} = \boldsymbol{\xi} = \begin{pmatrix} \xi_0 \\ \check{\boldsymbol{\xi}} \end{pmatrix}$$