ETH zürich

Lecture «Robot Dynamics»: Floating-base Systems

151-0851-00 V

lecture:	CAB G11	Tuesday 10:15 – 12:00, every week
exercise:	HG E1.2	Wednesday 8:15 – 10:00, according to schedule (about every 2nd week)

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ETH zürich

19.09.2017	Intro and Outline	Course Introduction; Recapitulation Position, Linear Velocity			
26.09.2017	Kinematics 1	Rotation and Angular Velocity; Rigid Body Formulation, Transformation	26.09.2017	Exercise 1a	Kinematics Modeling the ABB arm
03.10.2017	Kinematics 2	Kinematics of Systems of Bodies; Jacobians	03.10.2017	Exercise 1b	Differential Kinematics of the ABB arm
10.10.2017	Kinematics 3	Kinematic Control Methods: Inverse Differential Kinematics, Inverse Kinematics; Rotation Error; Multi-task Control	10.10.2017	Exercise 1c	Kinematic Control of the ABB Arm
17.10.2017	Dynamics L1	Multi-body Dynamics	17.10.2017	Exercise 2a	Dynamic Modeling of the ABB Arm
24.10.2017	Dynamics L2	Floating Base Dynamics	24.10.2017		
31.10.2017	Dynamics L3	Dynamic Model Based Control Methods	31.10.2017	Exercise 2b	Dynamic Control Methods Applied to the ABB arm
07.11.2017	Legged Robot	Dynamic Modeling of Legged Robots & Control	07.11.2017	Exercise 3	Legged robot
14.11.2017	Case Studies 1	Legged Robotics Case Study	14.11.2017		
21.11.2017	Rotorcraft	Dynamic Modeling of Rotorcraft & Control	21.11.2017	Exercise 4	Modeling and Control of Multicopter
28.11.2017	Case Studies 2	Rotor Craft Case Study	28.11.2017		
05.12.2017	Fixed-wing	Dynamic Modeling of Fixed-wing & Control	05.12.2017	Exercise 5	Fixed-wing Control and Simulation
12.12.2017	Case Studies 3	Fixed-wing Case Study (Solar-powered UAVs - AtlantikSolar, Vertical Take-off and Landing UAVs – Wingtra)			
19.12.2017	Summery and Outlook	Summery; Wrap-up; Exam			

Recapitulation of Introduction to Dynamics

- Description of “cause of motion”
  - Input $\boldsymbol{\tau}$ Force/Torque acting on system
  - Output $\ddot{\mathbf{q}}$ Motion of the system
- 3 methods to get the EoM
  - Newton-Euler: Free cut and conservation of impulse & angular mome
  - Projected Newton-Euler (generalized coordinates)
  - Lagrange II (energy)

$$\mathbf{M}(\mathbf{q})\ddot{\mathbf{q}}+\mathbf{b}(\mathbf{q},\dot{\mathbf{q}})+\mathbf{g}(\mathbf{q})=\mathbf{S}^T\boldsymbol{\tau}+\mathbf{J}_c^T\mathbf{F}_c$$

$\ddot{\mathbf{q}}$	Generalized coordinates
$\mathbf{M}(\mathbf{q})$	Mass matrix
$\mathbf{b}(\mathbf{q},\dot{\mathbf{q}})$	Centrifugal and Coriolis forces
$\mathbf{g}(\mathbf{q})$	Gravity forces
$\boldsymbol{\tau}$	Generalized forces
$\mathbf{S}_\tau$	Selection matrix/Jacobian
$\mathbf{F}_c$	External forces
$\mathbf{J}_c$	Contact Jacobian

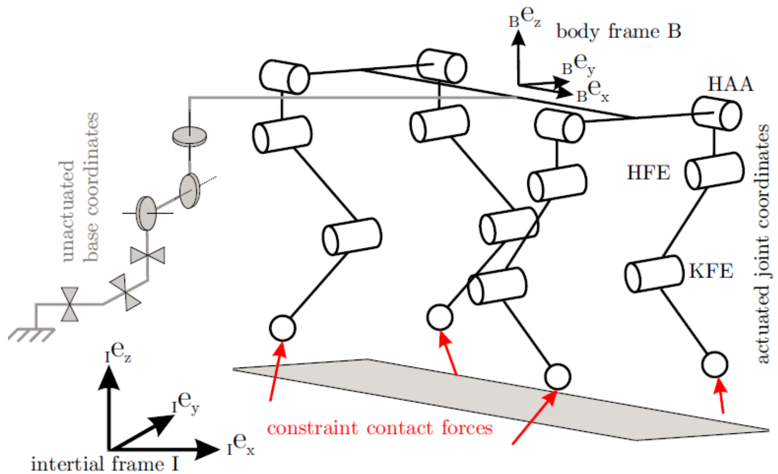
- Introduction to dynamics of floating base systems
  - External forces

Floating Base Systems
Kinematics

- Generalized coordinates

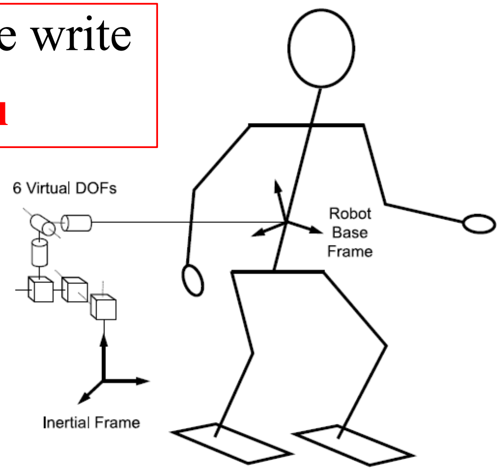
$$\mathbf{q} = \begin{pmatrix} \mathbf{q}_b \\ \mathbf{q}_j \end{pmatrix} \quad \text{with} \quad \mathbf{q}_b = \begin{pmatrix} \mathbf{q}_{bP} \\ \mathbf{q}_{bR} \end{pmatrix} \in \mathbb{R}^3 \times SO(3)$$

- Generalized velocities and accelerations?
  - Time derivatives $\dot{\mathbf{q}}, \ddot{\mathbf{q}}$ depend on parameterization
  - Often $\mathbf{u} = \begin{pmatrix} {}_I\mathbf{v}_B \\ \boxed{B}\boldsymbol{\omega}_{IB} \\ \dot{\varphi}_1 \\ \vdots \\ \dot{\varphi}_{n_j} \end{pmatrix} \in \mathbb{R}^{6+n_j} = \mathbb{R}^{n_u}$ $\quad \dot{\mathbf{u}} = \begin{pmatrix} {}_I\mathbf{a}_B \\ \boxed{B}\boldsymbol{\psi}_{IB} \\ \ddot{\varphi}_1 \\ \vdots \\ \ddot{\varphi}_{n_j} \end{pmatrix} \in \mathbb{R}^{6+n_j}$
  - Linear mapping $\mathbf{u} = \mathbf{E}_{fb} \cdot \dot{\mathbf{q}}$, with $\mathbf{E}_{fb} = \begin{bmatrix} \mathbb{I}_{3\times3} & 0 & 0 \\ 0 & \mathbf{E}_{\chi_R} & 0 \\ 0 & 0 & \mathbb{I}_{n_j \times n_j} \end{bmatrix}$



(a) Quadruped

Very often, people write $\dot{\mathbf{q}}$ but they mean $\mathbf{u}$



(b) Humanoid

Floating Base Systems
Differential kinematics

- Position of an arbitrary point on the robot

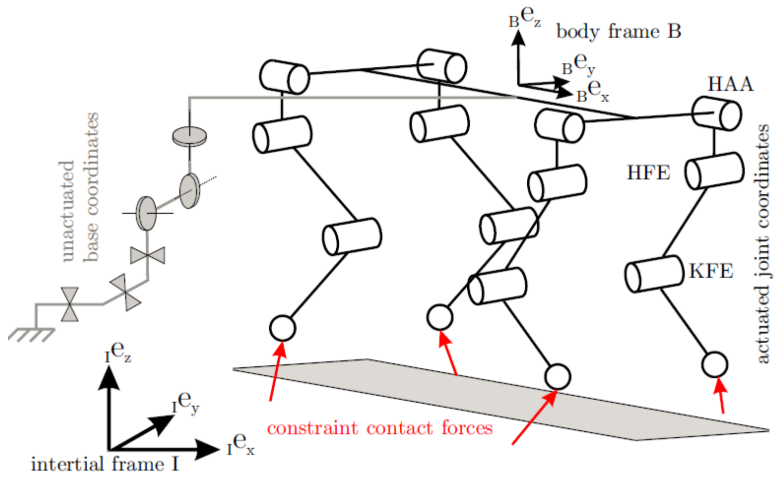
$$_{\mathcal{I}}\mathbf{r}_{IQ}(\mathbf{q}) = \underbrace{\boxed{_{\mathcal{I}}\mathbf{r}_{IB}(\mathbf{q})}}_{_{\mathcal{I}}\mathbf{r}_{IB}(\mathbf{q}_b)} + \underbrace{\boxed{\mathbf{C}_{\mathcal{IB}}(\mathbf{q})}}_{\mathbf{C}_{\mathcal{IB}}(\mathbf{q}_b)} \cdot \underbrace{\boxed{_{\mathcal{B}}\mathbf{r}_{BQ}(\mathbf{q})}}_{_{\mathcal{B}}\mathbf{r}_{BQ}(\mathbf{q}_j)}$$

- Velocity of this point

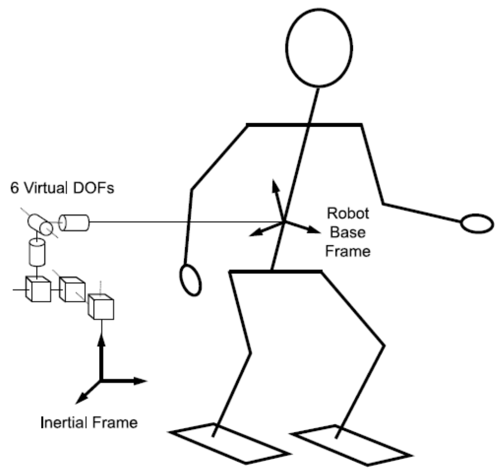
$$\begin{aligned}
_{\mathcal{I}}\mathbf{v}_Q &= {_{\mathcal{I}}\mathbf{v}_B} + \dot{\mathbf{C}}_{\mathcal{IB}} \cdot {_{\mathcal{B}}\mathbf{r}_{BQ}} + \mathbf{C}_{\mathcal{IB}} \cdot {_{\mathcal{B}}\dot{\mathbf{r}}_{BQ}} \\
&= {_{\mathcal{I}}\mathbf{v}_B} + \mathbf{C}_{\mathcal{IB}} \cdot [_{\mathcal{B}}\boldsymbol{\omega}_{\mathcal{IB}}]_\times \cdot {_{\mathcal{B}}\mathbf{r}_{BQ}} + \mathbf{C}_{\mathcal{IB}} \cdot {_{\mathcal{B}}\dot{\mathbf{r}}_{BQ}} \\
&= {_{\mathcal{I}}\mathbf{v}_B} - \mathbf{C}_{\mathcal{IB}} \cdot [_{\mathcal{B}}\mathbf{r}_{BQ}]_\times \cdot {_{\mathcal{B}}\boldsymbol{\omega}_{\mathcal{IB}}} + \mathbf{C}_{\mathcal{IB}} \cdot {_{\mathcal{B}}\dot{\mathbf{r}}_{BQ}} \\
&= {_{\mathcal{I}}\mathbf{v}_B} - \mathbf{C}_{\mathcal{IB}} \cdot [_{\mathcal{B}}\mathbf{r}_{BQ}]_\times \cdot {_{\mathcal{B}}\boldsymbol{\omega}_{\mathcal{IB}}} + \mathbf{C}_{\mathcal{IB}} \cdot {_{\mathcal{B}}\mathbf{J}_{P_{q_j}}}(\mathbf{q}_j) \cdot \dot{\mathbf{q}}_j \\
&= \underbrace{\boxed{\begin{bmatrix} \mathbb{I}_{3\times 3} & -\mathbf{C}_{\mathcal{IB}} \cdot [_{\mathcal{B}}\mathbf{r}_{BQ}]_\times & \mathbf{C}_{\mathcal{IB}} \cdot {_{\mathcal{B}}\mathbf{J}_{P_{q_j}}}(\mathbf{q}_j) \end{bmatrix}}}_{_{\mathcal{I}}\mathbf{J}_Q(\mathbf{q})} \cdot \mathbf{u}
\end{aligned}$$

with $\mathbf{u} = \begin{pmatrix} _{\mathcal{I}}\mathbf{v}_B \\ _{\mathcal{B}}\boldsymbol{\omega}_{IB} \\ \dot{\varphi}_1 \\ \vdots \\ \dot{\varphi}_{n_j} \end{pmatrix}$

$$[_{\mathcal{B}}\boldsymbol{\omega}_{\mathcal{IB}}]_\times = \mathbf{C}_{\mathcal{BI}}[_{\mathcal{I}}\boldsymbol{\omega}_{\mathcal{IB}}]_\times \mathbf{C}_{\mathcal{BI}}^T$$
$$= \mathbf{C}_{\mathcal{IB}}^T \dot{\mathbf{C}}_{\mathcal{IB}} \mathbf{C}_{\mathcal{IB}}^T \mathbf{C}_{\mathcal{IB}} = \mathbf{C}_{\mathcal{IB}}^T \dot{\mathbf{C}}_{\mathcal{IB}}$$



(a) Quadruped



(b) Humanoid

Contact Constraints

- A contact point C_i is not allowed to move:

$$_{\mathcal{I}}\mathbf{r}_{IC_i} = const, \quad {_{\mathcal{I}}}\dot{\mathbf{r}}_{IC_i} = {_{\mathcal{I}}}\ddot{\mathbf{r}}_{IC_i} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

- Constraint as a function of generalized coordinates:

$$_{\mathcal{I}}\mathbf{J}_{C_i}\mathbf{u} = \mathbf{0}, \qquad {_{\mathcal{I}}}\mathbf{J}_{C_i}\dot{\mathbf{u}} + {_{\mathcal{I}}}\dot{\mathbf{J}}_{C_i}\mathbf{u} = \mathbf{0}$$

- Stack of constraints

$$\mathbf{J}_c = \begin{bmatrix} \mathbf{J}_{C_1} \\ \vdots \\ \mathbf{J}_{C_{n_c}} \end{bmatrix} \in \mathbb{R}^{3n_c \times n_n}$$

Last time: Null-space motion

- Remember:

$$\mathbf{0} = \dot{\mathbf{r}}_c = \mathbf{J}_c \dot{\mathbf{q}} \quad \longrightarrow \quad \dot{\mathbf{q}} = \mathbf{J}_c^+ \mathbf{0} + \mathbf{N}_c \dot{\mathbf{q}}_0 = \mathbf{N}_c \dot{\mathbf{q}}_0$$

$$\mathbf{0} = \ddot{\mathbf{r}}_c = \mathbf{J}_c \ddot{\mathbf{q}} + \dot{\mathbf{J}}_c \dot{\mathbf{q}} \quad \longrightarrow \quad \ddot{\mathbf{q}} = \mathbf{J}_c^+ \left(-\dot{\mathbf{J}}_c \dot{\mathbf{q}} \right) + \mathbf{N}_c \ddot{\mathbf{q}}_0$$

- The system can be moved without violating the contact constraints!

- !! However, the base is unactuated !!
  - Which ones can be ACTIVELY controlled?

Properties of Contact Jacobian

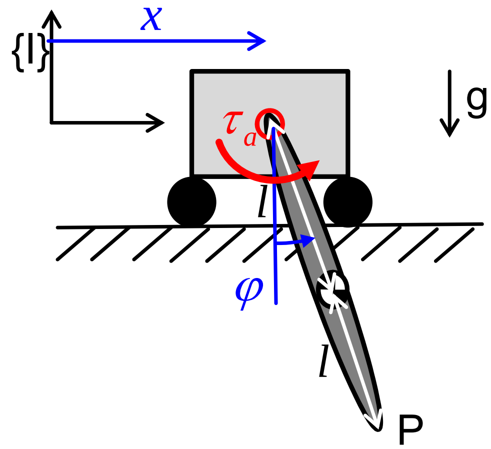
- Contact Jacobian tells us, how a system can move.
  - Separate stacked Jacobian $\mathbf{J}_c = \left[\boxed{\mathbf{J}_{c,b}} \quad \mathbf{J}_{c,j}\right] = \left[\frac{\partial \mathbf{r}_c}{\partial \mathbf{q}_b} \quad \frac{\partial \mathbf{r}_c}{\partial \mathbf{q}_j}\right] \in \mathbb{R}^{n_c \times (n_b + n_j)}$

    relation between base motion and constraints
  - Base is fully controllable if $rank(\mathbf{J}_{c,b}) = 6$
  - Nr of kinematic constraints for joint actuators: $rank(\mathbf{J}_c)$ - $rank(\mathbf{J}_{c,b})$
- Generalized coordinates DON'T correspond to the degrees of freedom
  - Contact constraints!
- Minimal coordinates (= correspond to degrees of freedom)
  - Require to switch the set of coordinates depending on contact state (=> never used)

Stupid, simple example
Cart pendulum

$$r_P = \begin{pmatrix} x + 2l\sin(\varphi) \\ -2l\cos(\varphi) \end{pmatrix}$$

$$\mathbf{q} = \begin{pmatrix} q_b \\ q_j \end{pmatrix} = \begin{pmatrix} x \\ \varphi \end{pmatrix}$$



- Analyse the kinematic constraints of this example
  - 1) P cannot move at all

$$\dot{\mathbf{r}}_P = \begin{pmatrix} \dot{x} + \dot{\varphi}2l\cos(\varphi) \\ \dot{\varphi}2l\sin(\varphi) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \mathbf{J}_c = \begin{bmatrix} \mathbf{J}_{c,b} & \mathbf{J}_{c,j} \end{bmatrix} = \begin{bmatrix} 1 & 2l\cos(\varphi) \\ 0 & 2l\sin(\varphi) \end{bmatrix}$$

$$rank(\mathbf{J}_c) = 2$$
$$rank(\mathbf{J}_{c,b}) = 1$$

  - 2) P can only move horizontally

$$\dot{y}_P = \dot{\varphi}2l\sin(\varphi) = 0 \Rightarrow \mathbf{J}_c = \begin{bmatrix} \mathbf{J}_{c,b} & \mathbf{J}_{c,j} \end{bmatrix} = \begin{bmatrix} 0 & 2l\sin(\varphi) \end{bmatrix}$$

$$rank(\mathbf{J}_c) = 1$$
$$rank(\mathbf{J}_{c,b}) = 0$$

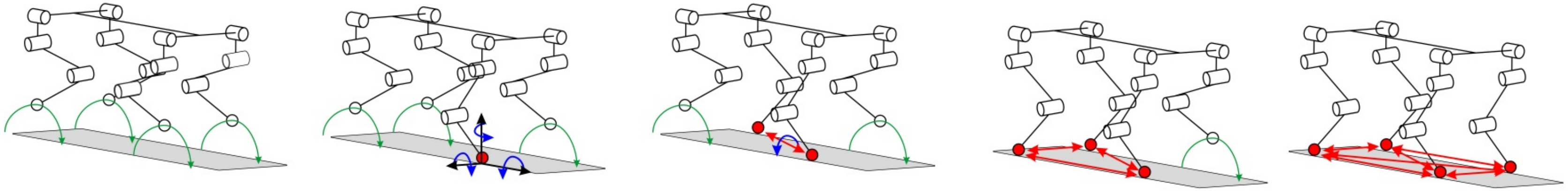
  - 3) P can only move vertically

$$\dot{x}_P = \dot{x} + \dot{\varphi}2l\cos(\varphi) = 0 \Rightarrow \mathbf{J}_c = \begin{bmatrix} \mathbf{J}_{c,b} & \mathbf{J}_{c,j} \end{bmatrix} = \begin{bmatrix} 1 & 2l\cos(\varphi) \end{bmatrix}$$

$$rank(\mathbf{J}_c) = 1$$
$$rank(\mathbf{J}_{c,b}) = 1$$

Quadrupedal Robot with Point Feet

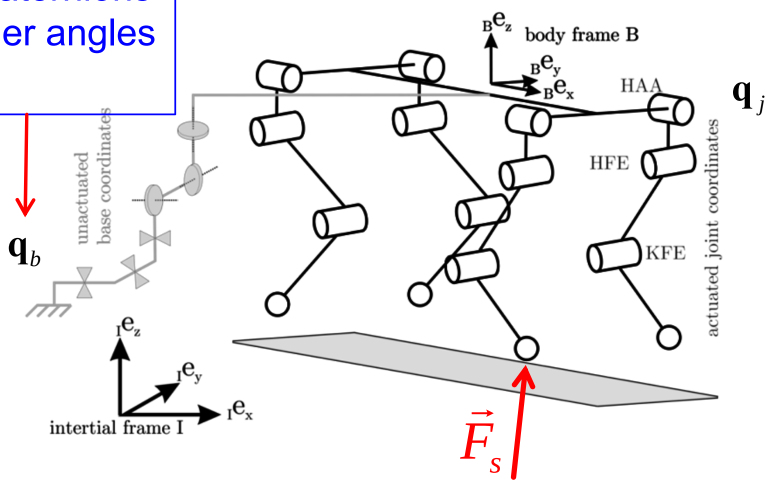
- Floating base system with 12 actuated joint and 6 base coordinates (18DoF)



Total constraints	0	3	6	9	12
Internal constraints	0	0	1	3	6
Uncontrollable DoFs	6	3	1	0	0

Dynamics of Floating Base Systems

- Quaternions
- Euler angles
- …



$$\mathbf{q} = \begin{pmatrix} \boxed{\mathbf{q}_b} \\ \mathbf{q}_j \end{pmatrix} \begin{matrix} \text{Un-actuated base} \\ \text{Actuated joints} \end{matrix}$$

- EoM from last time

$$\mathbf{M}\ddot{\mathbf{q}}_j + \mathbf{b} + \mathbf{g} = \boldsymbol{\tau}$$

- Not all joint are actuated

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{b} + \mathbf{g} = \mathbf{S}^T\boldsymbol{\tau}$$

  - Selection matrix of actuated joints

$$\mathbf{S} = \begin{bmatrix} \mathbf{0}_{n\times 6} & \mathbf{I}_{n\times n} \end{bmatrix} \qquad \mathbf{q}_j = \mathbf{S}\mathbf{q}$$

- Contact force acting on system

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{b} + \mathbf{g} = \mathbf{S}^T\boldsymbol{\tau} + \mathbf{J}_s^T\mathbf{F}_{s,\text{ acting on system}}$$

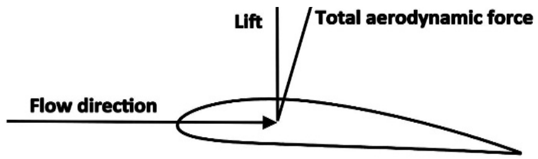
$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{b} + \mathbf{g} + \mathbf{J}_s^T\mathbf{F}_{s,\text{ exerted by robot}} = \mathbf{S}^T\boldsymbol{\tau}$$

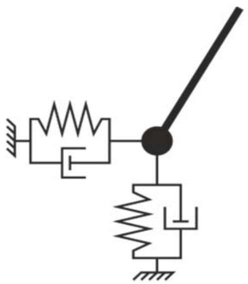
Manipulator:	interaction forces at end-effector
Legged robot:	ground contact forces
UAV:	lift force

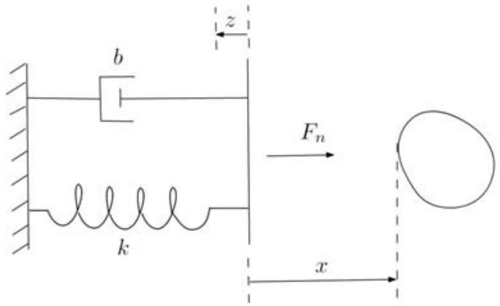
*Note: for simplicity we don't use here **u** but only time derivatives of **q***

External Forces
Some notes

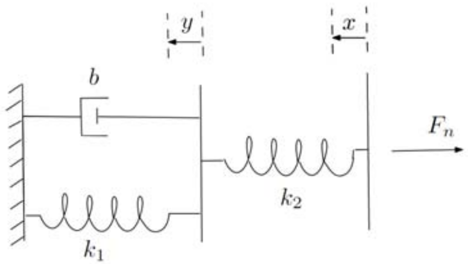
- External forces from force elements or actuator
  - Aerodynamics $F_s = \frac{1}{2}\rho c_v A c_L$
  - Contact
    - Simple solution: soft contact model $\mathbf{F}_c = k_p(\mathbf{r}_c - \mathbf{r}_{c0}) + k_d\dot{\mathbf{r}}_c$
    - no negative force $F_n = \begin{cases} 0 & \text{if } x > z \\ max(0, kz + b\dot{z}) & \text{if } x = z \end{cases}$







Linear S-D 1



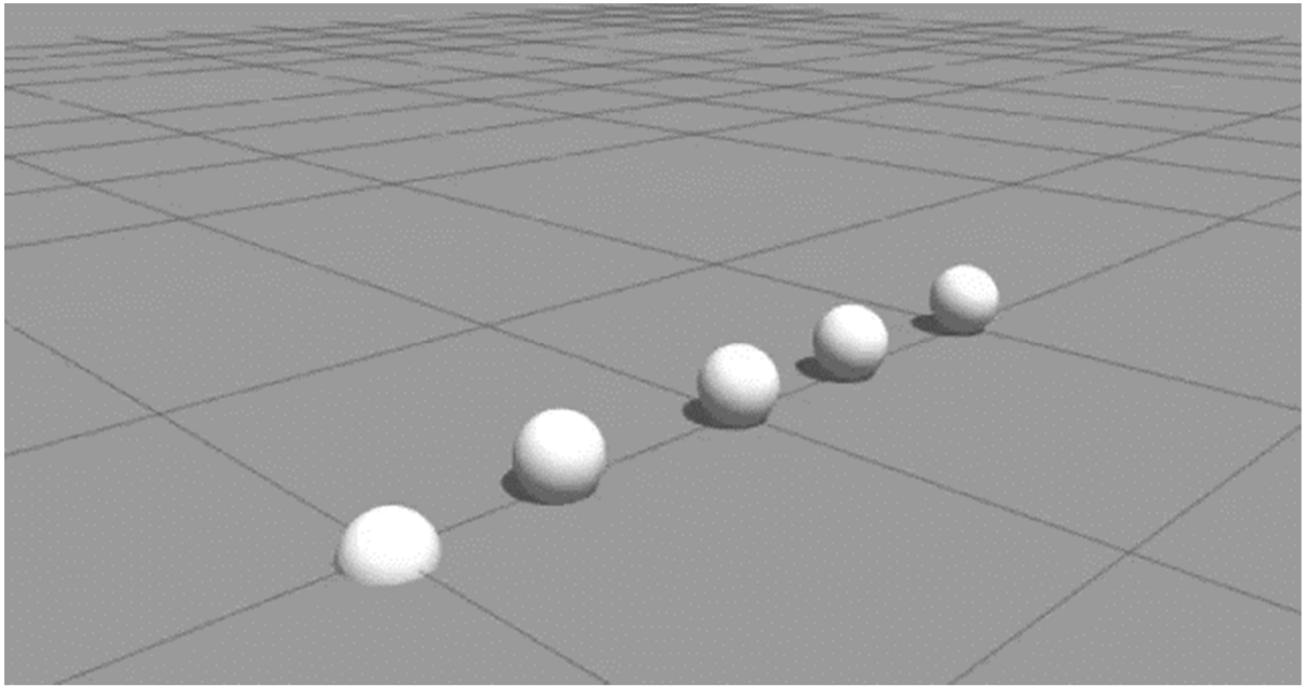
Linear S-D 2

$F_n = bx^n\dot{x} + kx^n$

Nonlinear

Soft Contact

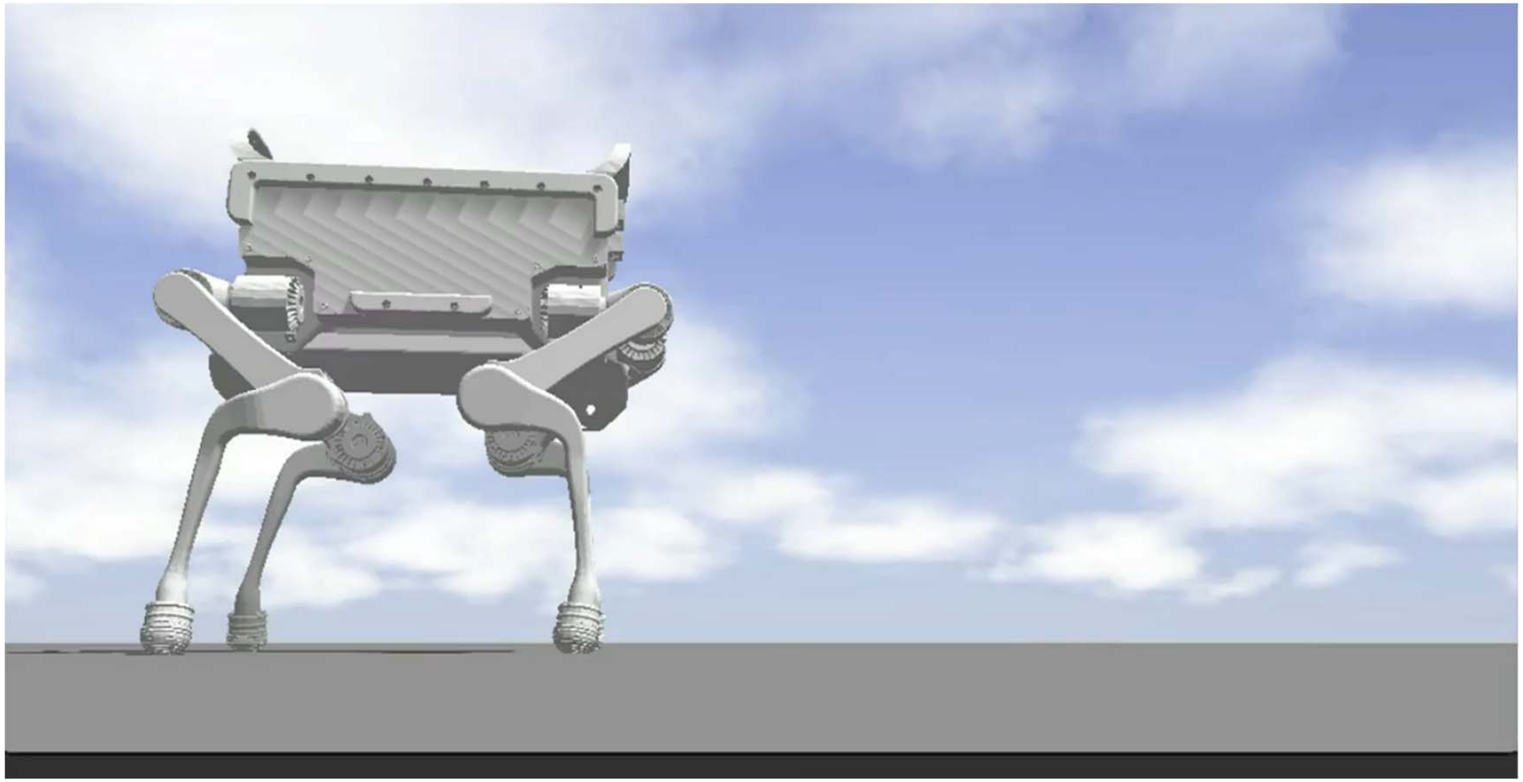
Physical accuracy vs. numerical stability?



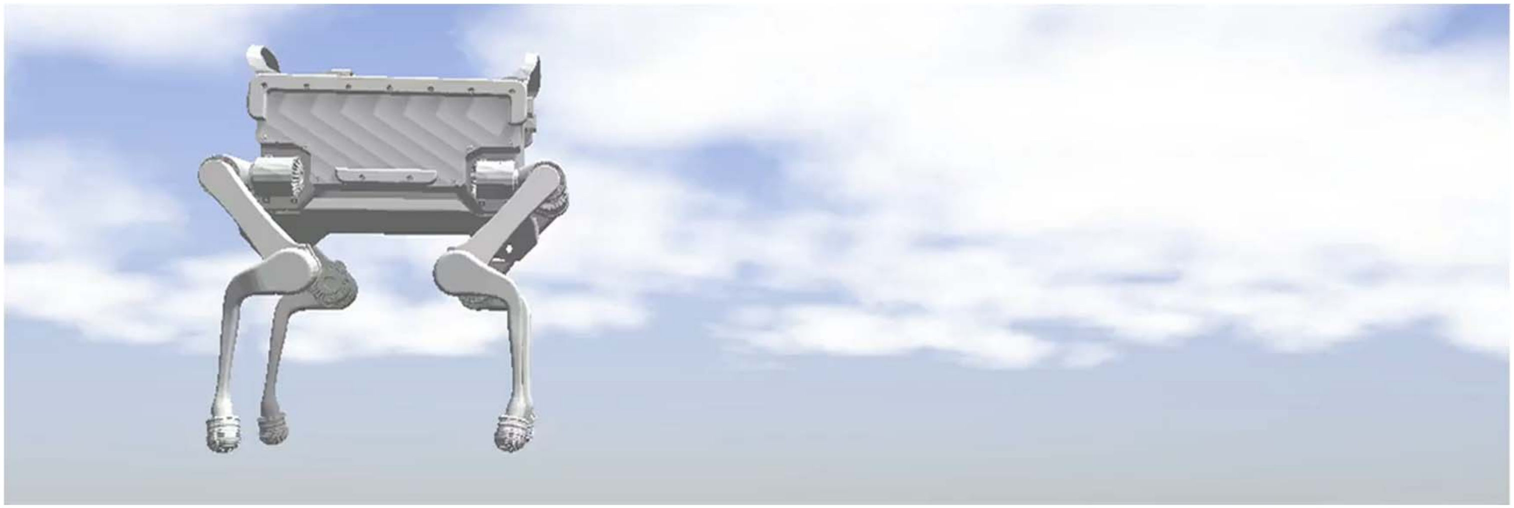
Soft Contact
Physical accuracy vs. numerical stability?

- Stiff equation of motion
  - Small time steps
  - Can lead to instability
- Contact behavior strongly depends on the robot parameters/configuration
- Contact parameters are not selected as physical parameters but as numerical
  - Trade-off stability ⇔ accuracy

ANYmal in Simulation



ANYmal in Simulation



Hard Contact

- External forces from constraints
  - Equation of motion $$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{b} + \mathbf{g} + \mathbf{J}_c^T \mathbf{F}_c = \mathbf{S}^T \boldsymbol{\tau} \qquad (1)$$
  - Contact constraint $$\dot{\mathbf{r}}_c = \mathbf{J}_c \dot{\mathbf{q}} = \mathbf{0} \Rightarrow \ddot{\mathbf{r}}_c = \mathbf{J}_c \ddot{\mathbf{q}} + \dot{\mathbf{J}}_c \dot{\mathbf{q}} = \mathbf{0} \qquad (2)$$
  - Substitute $\ddot{q}$ in (2) from (1) $$\ddot{\mathbf{r}}_c = \mathbf{J}_c \mathbf{M}^{-1}\left(\mathbf{S}^T \boldsymbol{\tau} - \left(\mathbf{b} + \mathbf{g} + \mathbf{J}_c^T \mathbf{F}_c\right)\right) + \dot{\mathbf{J}}_c \dot{\mathbf{q}} = \mathbf{0} \qquad (3)$$
  - Solve (3) for contact force $$\mathbf{F}_c = \left(\mathbf{J}_c \mathbf{M}^{-1} \mathbf{J}_c^T\right)^{-1} \left(\mathbf{J}_c \mathbf{M}^{-1}\left(\mathbf{S}^T \boldsymbol{\tau} - (\mathbf{b} + \mathbf{g})\right) + \dot{\mathbf{J}}_c \dot{\mathbf{q}}\right)$$
  - Back-substitute in (1), replace $\dot{\mathbf{J}}_s \dot{\mathbf{q}} = -\mathbf{J}_s \ddot{\mathbf{q}}$ and use support null-space projection $$\mathbf{N}_c = \mathbf{I} - \mathbf{M}^{-1} \mathbf{J}_c^T \left(\mathbf{J}_c \mathbf{M}^{-1} \mathbf{J}_c^T\right)^{-1} \mathbf{J}_c \qquad \boxed{\mathbf{J}_c \mathbf{N}_c = \mathbf{0}}$$
- Support consistent dynamics $$\boxed{\mathbf{N}_c^T \mathbf{M}\ddot{\mathbf{q}} + \mathbf{N}_c^T (\mathbf{b} + \mathbf{g}) = \mathbf{N}_c^T \mathbf{S}^T \boldsymbol{\tau}}$$
  -

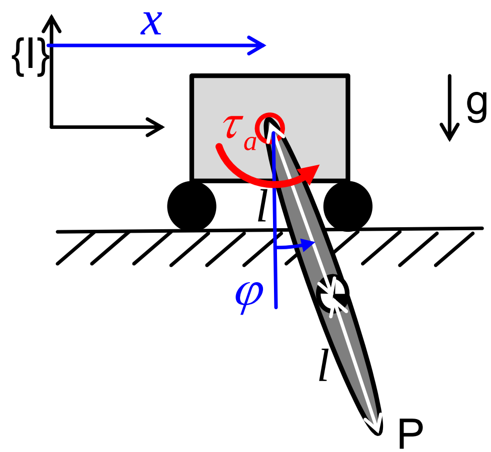
This is only a projector,
… we can use other ones

Simple example
Cart Pendulum

$$\boxed{\mathbf{N}_c^T\mathbf{M}\ddot{\mathbf{q}} + \mathbf{N}_c^T\left(\mathbf{b}+\mathbf{g}\right) = \mathbf{N}_c^T\mathbf{S}^T\boldsymbol{\tau}}$$

$$\mathbf{q} = \begin{pmatrix} q_b \\ q_j \end{pmatrix} = \begin{pmatrix} x \\ \varphi \end{pmatrix}$$

$$\underbrace{\begin{bmatrix} m_c + m_p & l m_p \cos(\varphi) \\ l m_p \cos(\varphi) & m_p l^2 + \theta_p \end{bmatrix}}_{\mathbf{M}} \ddot{\mathbf{q}} + \underbrace{\begin{pmatrix} -\dot{\varphi}^2 l m_p \sin(\varphi) \\ 0 \end{pmatrix}}_{\mathbf{b}} + \underbrace{\begin{pmatrix} 0 \\ m_p g l \sin(\varphi) \end{pmatrix}}_{\mathbf{g}} = \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{\mathbf{S}^T} \tau_a$$



- Contact Jacobian: $\mathbf{J}_c = \left[\mathbf{J}_{c,b} \quad \mathbf{J}_{c,j}\right] = \begin{bmatrix} 1 & 2l\cos(\varphi) \\ 0 & 2l\sin(\varphi) \end{bmatrix}$

1) both constraints active:

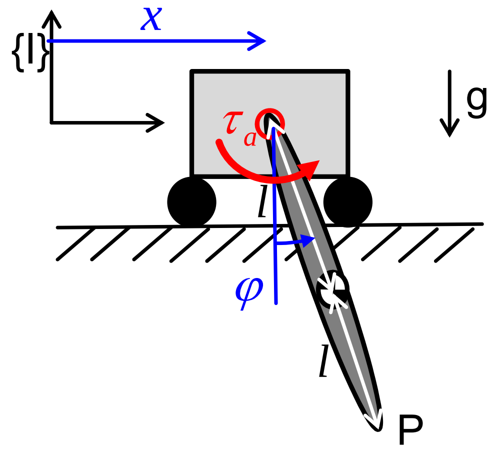
- No controllable subspace $\mathcal{N}(\mathbf{J}_c) = 0$
- => constraint define the «motion», τ_a can be freely choosen and only changes $\mathbf{F}_c$

Simple example
Cart Pendulum

$$\boxed{\mathbf{N}_c^T \mathbf{M}\ddot{\mathbf{q}} + \mathbf{N}_c^T\left(\mathbf{b}+\mathbf{g}\right) = \mathbf{N}_c^T \mathbf{S}^T \boldsymbol{\tau}}$$

$$\mathbf{q} = \begin{pmatrix} q_b \\ q_j \end{pmatrix} = \begin{pmatrix} x \\ \varphi \end{pmatrix}$$

$$\underbrace{\begin{bmatrix} m_c + m_p & l m_p \cos(\varphi) \\ l m_p \cos(\varphi) & m_p l^2 + \theta_p \end{bmatrix}}_{\mathbf{M}} \ddot{\mathbf{q}} + \underbrace{\begin{pmatrix} -\dot{\varphi}^2 l m_p \sin(\varphi) \\ 0 \end{pmatrix}}_{\mathbf{b}} + \underbrace{\begin{pmatrix} 0 \\ m_p g l \sin(\varphi) \end{pmatrix}}_{\mathbf{g}} = \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{\mathbf{S}^T} \tau_a$$



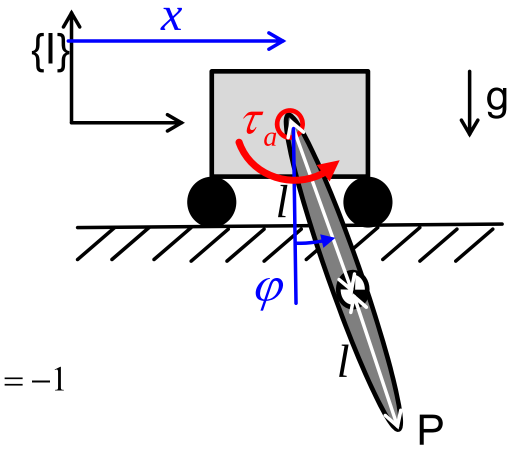
- Contact Jacobian: $\mathbf{J}_c = \begin{bmatrix} \mathbf{J}_{c,b} & \mathbf{J}_{c,j} \end{bmatrix} = \begin{bmatrix} \cancel{1} & \cancel{2l\cos(\varphi)} \\ 0 & 2l\sin(\varphi) \end{bmatrix}$
- 2) vertical motion locked: $\mathbf{N}_c = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \mathbf{N}_c^T \mathbf{S}^T = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0$
  -
  - first line corresponds to support consistent dynamics
  - Torque has no influence on motion
  - Torque can be used to modify the constraint force $\mathcal{N}\left(\mathbf{N}_c^T \mathbf{S}^T\right) = \mathcal{N}(0) = 1$

Simple example
Cart Pendulum

$$\boxed{\mathbf{N}_c^T\mathbf{M}\ddot{\mathbf{q}} + \mathbf{N}_c^T\left(\mathbf{b}+\mathbf{g}\right) = \mathbf{N}_c^T\mathbf{S}^T\boldsymbol{\tau}}$$

$$\mathbf{q} = \begin{pmatrix} q_b \\ q_j \end{pmatrix} = \begin{pmatrix} x \\ \varphi \end{pmatrix}$$

$$\underbrace{\begin{bmatrix} m_c + m_p & l m_p \cos(\varphi) \\ l m_p \cos(\varphi) & m_p l^2 + \theta_p \end{bmatrix}}_{\mathbf{M}} \ddot{\mathbf{q}} + \underbrace{\begin{pmatrix} -\dot{\varphi}^2 l m_p \sin(\varphi) \\ 0 \end{pmatrix}}_{\mathbf{b}} + \underbrace{\begin{pmatrix} 0 \\ m_p g l \sin(\varphi) \end{pmatrix}}_{\mathbf{g}} = \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{\mathbf{S}^T} \tau_a$$



- Contact Jacobian: $\mathbf{J}_c = \begin{bmatrix} \mathbf{J}_{c,b} & \mathbf{J}_{c,j} \end{bmatrix} = \begin{bmatrix} 1 & 2l\cos(\varphi) \\ \cancel{0} & \cancel{2l\sin(\varphi)} \end{bmatrix}$
- 2) horizontal motion locked: $\mathbf{N}_c = \begin{bmatrix} 2l\cos(\varphi) \\ -1 \end{bmatrix}$ $\quad \mathbf{N}_c^T\mathbf{S}^T = \begin{bmatrix} 2l\cos(\varphi) & -1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = -1$
  -
  - first line corresponds to support consistent dynamics
  - There is no null-space that can change the constraint force without changing the motion

Some more insights into the EoM

$$\begin{bmatrix} \mathbf{M}_{bb} & \mathbf{M}_{bj} \\ \mathbf{M}_{jb} & \mathbf{M}_{jj} \end{bmatrix} \begin{pmatrix} \dot{\mathbf{u}}_b \\ \dot{\mathbf{u}}_j \end{pmatrix} + \begin{pmatrix} \mathbf{b}_b \\ \mathbf{b}_j \end{pmatrix} + \begin{pmatrix} \mathbf{g}_b \\ \mathbf{g}_j \end{pmatrix} + \begin{bmatrix} \mathbf{J}_{c,b}^T \\ \mathbf{J}_{c,j}^T \end{bmatrix} \mathbf{F}_c = \begin{pmatrix} \mathbf{0} \\ \boldsymbol{\tau} \end{pmatrix}$$

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Contact Dynamics

- Impulse transfer at contact
  - Integration over a single point in time $$\int_{\{t_0\}} \left(\mathbf{M}\dot{\mathbf{u}} + \mathbf{b} + \mathbf{g} + \mathbf{J}_c^T \mathbf{F}_c - \mathbf{S}^T \boldsymbol{\tau}\right) \mathrm{d}t = \mathbf{M}\left(\mathbf{u}^+ - \mathbf{u}^-\right) + \mathbf{J}_c^T \mathcal{F}_c = \mathbf{0}$$
  - Post impact condition $$\dot{\mathbf{r}}_c^+ = \mathbf{J}_c \mathbf{u}^+ = \mathbf{0}$$
  - Impulsive force $$\mathcal{F}_c = \left(\mathbf{J}_c \mathbf{M}^{-1} \mathbf{J}_c^T\right)^{-1} \mathbf{J}_c \dot{\mathbf{q}}^- = \boldsymbol{\Lambda}_c \dot{\mathbf{r}}_c^-$$
    - End-effector inertia $$\boldsymbol{\Lambda}_c = \left(\mathbf{J}_c \mathbf{M}^{-1} \mathbf{J}_c^T\right)^{-1}$$
    - Change in generalized velocity $$\Delta\mathbf{u} = \mathbf{u}^+ - \mathbf{u}^- = -\mathbf{M}^{-1} \mathbf{J}_c^T \left(\mathbf{J}_c \mathbf{M}^{-1} \mathbf{J}_c^T\right)^{-1} \mathbf{J}_c \mathbf{u}^-$$
    - Post-impact velocity $$\mathbf{u}^+ = \left(\mathbb{I} - \mathbf{M}^{-1} \mathbf{J}_c^T \left(\mathbf{J}_c \mathbf{M}^{-1} \mathbf{J}_c^T\right)^{-1} \mathbf{J}_c\right) \mathbf{u}^- = \mathbf{N}_c \mathbf{u}^-$$
    - Energy loss $$\begin{aligned} E_{loss} = \Delta E_{kin} &= -\tfrac{1}{2} \Delta\mathbf{u}^T \mathbf{M} \Delta\mathbf{u} \\ &= -\tfrac{1}{2} \Delta\dot{\mathbf{r}}_c^T \boldsymbol{\Lambda}_c \Delta\dot{\mathbf{r}}_c = -\tfrac{1}{2} \dot{\mathbf{r}}_c^{-T} \boldsymbol{\Lambda}_c \dot{\mathbf{r}}_c^- \end{aligned}$$

Generalized vs Minimal coordinates

- Define minimal coordinates $\boldsymbol{\xi} = \boldsymbol{\xi}(\mathbf{q}), \qquad \mathbf{Q}_\xi := \frac{\partial \mathbf{q}}{\partial \boldsymbol{\xi}}$
- Virtual displacent are consistent with constraints $\delta \mathbf{q}_{\text{consistent}} = \frac{\partial \mathbf{q}}{\partial \boldsymbol{\xi}} \delta \boldsymbol{\xi} = \mathbf{Q}_\xi \delta \boldsymbol{\xi}.$
  - They produce no displacement at contacts

$$\begin{aligned} \delta \mathbf{r}_{s,\text{consistent}} &= \frac{\partial \mathbf{r}_s}{\partial \mathbf{q}} \delta \mathbf{q}_{\text{consistent}} = \mathbf{J}_s \mathbf{Q}_\xi \delta \boldsymbol{\xi} = \mathbf{0} \qquad \forall \boldsymbol{\xi} \\ \Rightarrow \mathbf{J}_s \mathbf{Q}_\xi &= \mathbf{0}. \end{aligned}$$

- (with this) Principle of virtual work

$$\begin{aligned} \delta \mathbf{q}_{\text{consistent}}^T \left(\mathbf{M}\ddot{\mathbf{q}} + \mathbf{h} + \mathbf{J}_s^T \mathbf{F}_s - \mathbf{S}^T \boldsymbol{\tau} \right) &= 0 \quad \forall \delta \mathbf{q}_{\text{consistent}} \\ \delta \xi^T \mathbf{Q}_\xi^T \left(\mathbf{M}\ddot{\mathbf{q}} + \mathbf{h} + \mathbf{J}_s^T \mathbf{F}_s - \mathbf{S}^T \boldsymbol{\tau} \right) &= 0 \quad \forall \delta \boldsymbol{\xi}. \end{aligned}$$

- Substituting $\ddot{\mathrm{q}} = \mathbf{Q}_\xi \ddot{\boldsymbol{\xi}} + \dot{\mathbf{Q}}_\xi \dot{\boldsymbol{\xi}}$ yields

$$\mathbf{Q}_\xi^T \mathbf{M} \mathbf{Q}_\xi \ddot{\boldsymbol{\xi}} + \mathbf{Q}_\xi^T \left(\mathbf{h} + \mathbf{M} \dot{\mathbf{Q}}_\xi \dot{\boldsymbol{\xi}} \right) = \mathbf{Q}_\xi^T \mathbf{S}^T \boldsymbol{\tau}$$