ETH zürich

Lecture «Robot Dynamics»: Kinematics 2

151-0851-00 V

lecture:	CAB G11	Tuesday 10:15 – 12:00, every week
exercise:	HG E1.2	Wednesday 8:15 – 10:00, according to schedule (about every 2nd week)

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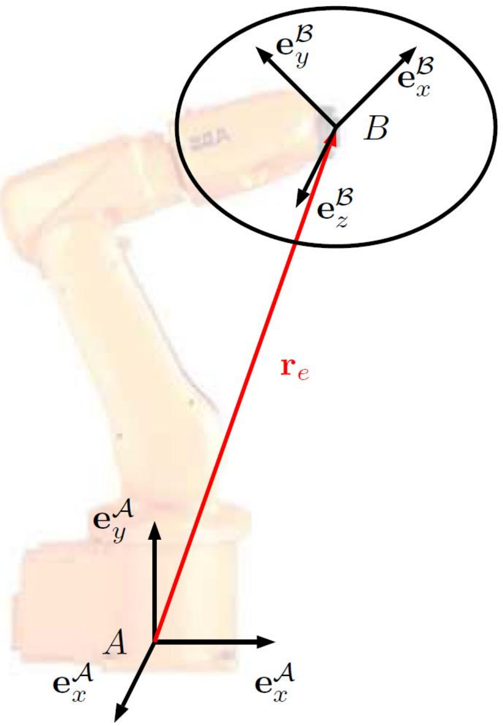
19.09.2017	Intro and Outline	Course Introduction; Recapitulation Position, Linear Velocity			
26.09.2017	Kinematics 1	Rotation and Angular Velocity; Rigid Body Formulation, Transformation	26.09.2017	Exercise 1a	Kinematics Modeling the ABB arm
03.10.2017	Kinematics 2	Kinematics of Systems of Bodies; Jacobians	03.10.2017	Exercise 1b	Differential Kinematics of the ABB arm
10.10.2017	Kinematics 3	Kinematic Control Methods: Inverse Differential Kinematics, Inverse Kinematics; Rotation Error; Multi-task Control	10.10.2017	Exercise 1c	Kinematic Control of the ABB Arm
17.10.2017	Dynamics L1	Multi-body Dynamics	17.10.2017	Exercise 2a	Dynamic Modeling of the ABB Arm
24.10.2017	Dynamics L2	Floating Base Dynamics	24.10.2017		
31.10.2017	Dynamics L3	Dynamic Model Based Control Methods	31.10.2017	Exercise 2b	Dynamic Control Methods Applied to the ABB arm
07.11.2017	Legged Robot	Dynamic Modeling of Legged Robots & Control	07.11.2017	Exercise 3	Legged robot
14.11.2017	Case Studies 1	Legged Robotics Case Study	14.11.2017		
21.11.2017	Rotorcraft	Dynamic Modeling of Rotorcraft & Control	21.11.2017	Exercise 4	Modeling and Control of Multicopter
28.11.2017	Case Studies 2	Rotor Craft Case Study	28.11.2017		
05.12.2017	Fixed-wing	Dynamic Modeling of Fixed-wing & Control	05.12.2017	Exercise 5	Fixed-wing Control and Simulation
12.12.2017	Case Studies 3	Fixed-wing Case Study (Solar-powered UAVs - AtlantikSolar, Vertical Take-off and Landing UAVs – Wingtra)			
19.12.2017	Summery and Outlook	Summery; Wrap-up; Exam			

Last Time: Position Parameterization

- Position vector: $\mathbf{r}_e = \mathbf{r}_e(\boldsymbol{\chi}) \in \mathbb{R}^3$
- Parameterization: $\boldsymbol{\chi}_P \in \mathbb{R}^3$
  - Cartesian $\boldsymbol{\chi}_{Pc} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$
  - Cylindrical coordinates $\boldsymbol{\chi}_{Pz} = \begin{pmatrix} \rho \\ \theta \\ z \end{pmatrix}$
  - Spherical coordinates $\boldsymbol{\chi}_{Ps} = \begin{pmatrix} r \\ \theta \\ \phi \end{pmatrix}$



- Relation between linear velocity and parameter differentiation $\dot{\mathbf{r}}_e = \frac{\partial \mathbf{r}_e}{\partial \boldsymbol{\chi}_P} \dot{\boldsymbol{\chi}}_P = \mathbf{E}_P \dot{\boldsymbol{\chi}}_P$

  … with the parameterization specific matrix $\mathbf{E}_P(\boldsymbol{\chi}_P) \in \mathbb{R}^{3\times 3}$

Rotation Parameterization

- Rotation matrix:
  - 3x3 = 9 parameters
  - Orthonormality = 6 constraints

$$\mathbf{C}_{\mathcal{AB}} = \begin{bmatrix} {}_{\mathcal{A}}\mathbf{e}_x^{\mathcal{B}} & {}_{\mathcal{A}}\mathbf{e}_y^{\mathcal{B}} & {}_{\mathcal{A}}\mathbf{e}_z^{\mathcal{B}} \end{bmatrix}$$

- Euler Angles
  - 3 parameters, singularity problem

$$\chi_{R,eulerZYX} = \begin{pmatrix} z \\ y \\ x \end{pmatrix}$$

- Angle Axis
  - 4 parameters, unitary constraint, singularity problem

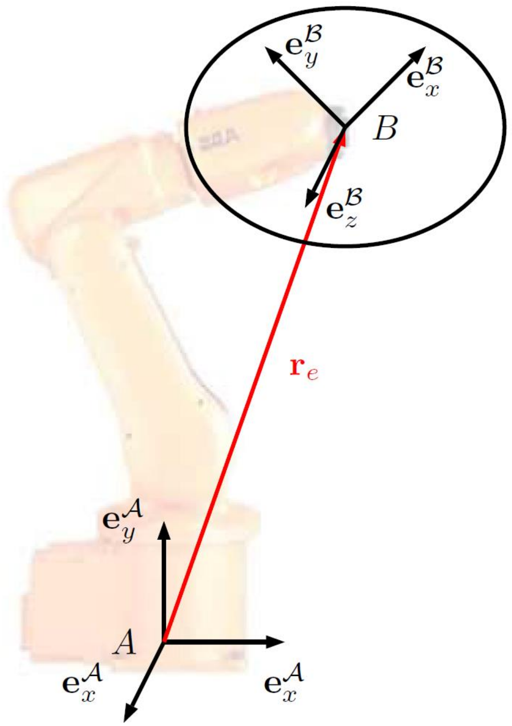
$$\chi_{R,AngleAxis} = \begin{pmatrix} \theta \\ \mathbf{n} \end{pmatrix}$$

- Rotation vector
  - 3 parameters, singularity problem

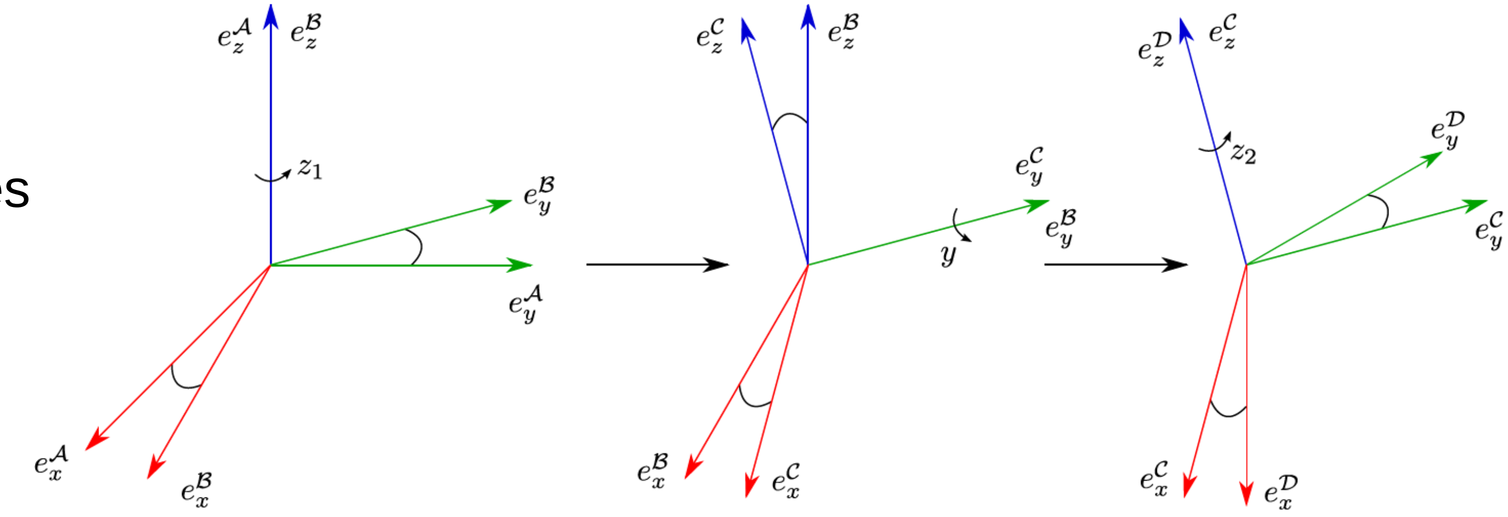
$$\chi_{R,rotvec} = \boldsymbol{\varphi} = \theta \mathbf{n}$$

- Quaternions
  - 4 parameters
  - no singularity

$$\chi_{R,quat} = \boldsymbol{\xi} = \begin{pmatrix} \xi_0 \\ \check{\boldsymbol{\xi}} \end{pmatrix}$$



Euler Angles
Consecutive elementary rotations



- Three elementary rotations
  - ZYZ and ZXZ: proper Euler angles
  - ZYX: Tait-Bryan angles
  - XYZ: Cardan angles

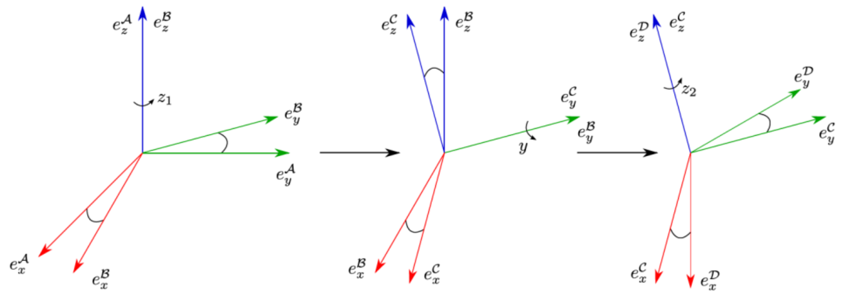
  - Example Z-Y-Z

$$\chi_{R,eulerZYZ} = \begin{pmatrix} z_1 \\ y \\ z_2 \end{pmatrix}$$

$$\mathbf{C}_{\mathcal{AD}} = \mathbf{C}_{\mathcal{AD}}\left(\chi_{R,eulerZYZ}\right) = \mathbf{C}_{\mathcal{AB}}\left(z_1\right)\mathbf{C}_{\mathcal{BC}}\left(y\right)\mathbf{C}_{\mathcal{CD}}\left(z_2\right)$$

From Euler Angles to Rotation Matrix

ZYZ example



$$\mathbf{C}_{\mathcal{AD}} = \mathbf{C}_{\mathcal{AD}}\left(\chi_{R,eulerZYZ}\right) = \mathbf{C}_{\mathcal{AB}}\left(z_1\right)\mathbf{C}_{\mathcal{BC}}\left(y\right)\mathbf{C}_{\mathcal{CD}}\left(z_2\right)$$

$$\mathbf{C}_{\mathcal{AD}} = \mathbf{C}_{\mathcal{AB}}(z_1)\mathbf{C}_{\mathcal{BC}}(y)\mathbf{C}_{\mathcal{CD}}(z_2) \quad \Rightarrow \quad {}_{\mathcal{A}}\mathbf{r} = \mathbf{C}_{\mathcal{AD}}\,{}_{\mathcal{D}}\mathbf{r}$$

$$= \begin{bmatrix} \cos z_1 & -\sin z_1 & 0 \\ \sin z_1 & \cos z_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos y & 0 & \sin y \\ 0 & 1 & 0 \\ -\sin y & 0 & \cos y \end{bmatrix} \begin{bmatrix} \cos z_2 & -\sin z_2 & 0 \\ \sin z_2 & \cos z_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} c_y c_{z_1} c_{z_2} - s_{z_1} s_{z_2} & -c_{z_2} s_{z_1} - c_y c_{z_1} s_{z_2} & c_{z_1} s_y \\ c_{z_1} s_{z_2} + c_y c_{z_2} s_{z_1} & c_{z_1} c_{z_2} - c_y s_{z_1} s_{z_2} & s_y s_{z_1} \\ -c_{z_2} s_y & s_y s_{z_2} & c_y \end{bmatrix}.$$

From Rotation Matrix to Euler Angles
ZYZ example

- A rotation matrix has the following form

$$\mathbf{C}_{\mathcal{AD}} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}$$

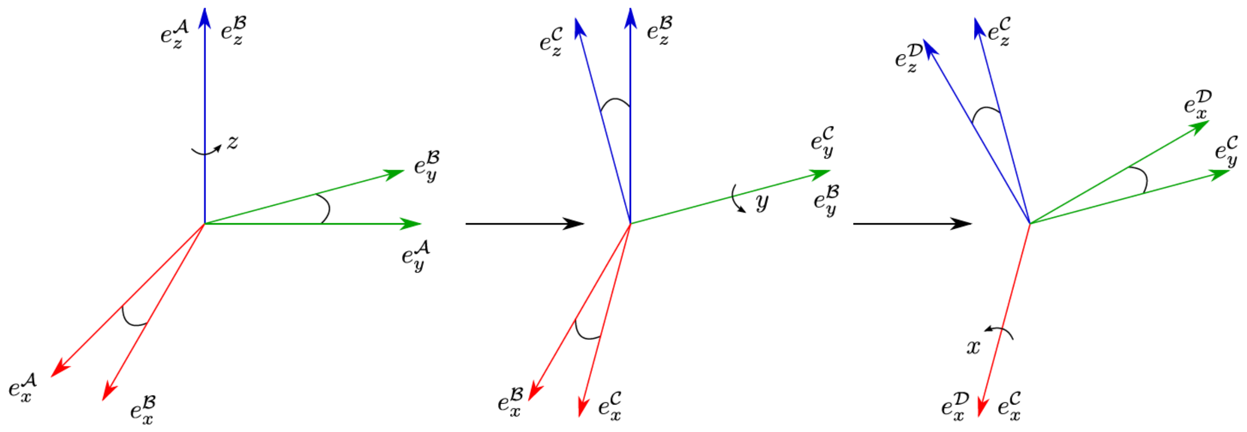
- As a function of ZYZ Euler Angles, we found

$$\mathbf{C}_{\mathcal{AD}} = \begin{bmatrix} c_y c_{z_1} c_{z_2} - s_{z_1} s_{z_2} & -c_{z_2} s_{z_1} - c_y c_{z_1} s_{z_2} & c_{z_1} s_y \\ c_{z_1} s_{z_2} + c_y c_{z_2} s_{z_1} & c_{z_1} c_{z_2} - c_y s_{z_1} s_{z_2} & s_y s_{z_1} \\ -c_{z_2} s_y & s_y s_{z_2} & c_y \end{bmatrix}$$

$$\boxed{\chi_{R,eulerZYZ} = \begin{pmatrix} z_1 \\ y \\ z_2 \end{pmatrix} = \begin{pmatrix} atan2\left(c_{23}, c_{13}\right) \\ atan2\left(\sqrt{c_{13}^2 + c_{23}^2}, c_{33}\right) \\ atan2\left(c_{32}, -c_{31}\right) \end{pmatrix}}$$

Atan2 function:
uses sign of both arguments to determine the correct quadrant

Euler Angles ⇔ Rotation Matrix
ZYX example



- Rotation parameters

$$\chi_{R,eulerZYX} = \begin{pmatrix} z \\ y \\ x \end{pmatrix}$$

- Rotation matrix from Euler Angles

$$\mathbf{C}_{\mathcal{AD}} = \mathbf{C}_{\mathcal{AB}}(z)\mathbf{C}_{\mathcal{BC}}(y)\mathbf{C}_{\mathcal{CD}}(x) \quad \Rightarrow \quad {}_{\mathcal{A}}\mathbf{r} = \mathbf{C}_{\mathcal{AD}}\,{}_{\mathcal{D}}\mathbf{r}$$

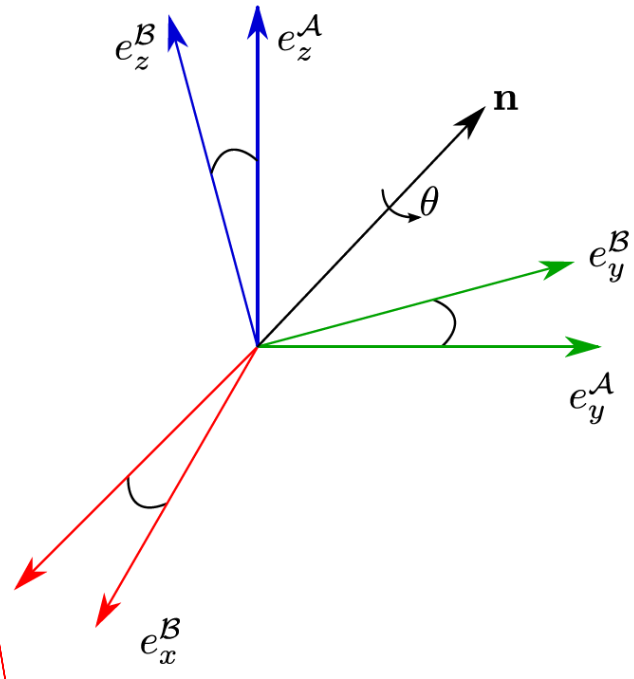
$$= \begin{bmatrix} \cos z & -\sin z & 0 \\ \sin z & \cos z & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos y & 0 & \sin y \\ 0 & 1 & 0 \\ -\sin y & 0 & \cos y \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos x & -\sin x \\ 0 & \sin x & \cos x \end{bmatrix}$$

$$= \begin{bmatrix} c_y c_z & c_z s_x s_y - c_x s_z & s_x s_z + c_x c_z s_y \\ c_y s_z & c_x c_z + s_x s_y s_z & c_x s_y s_z - c_z s_x \\ -s_y & c_y s_x & c_x c_y \end{bmatrix}.$$

- Euler Angles from Rotation matrix

$$\chi_{R,eulerZYX} = \begin{pmatrix} z \\ y \\ x \end{pmatrix} = \begin{pmatrix} atan2\left(c_{21}, c_{11}\right) \\ atan2\left(-c_{31}, \sqrt{c_{32}^2 + c_{33}^2}\right) \\ atan2\left(c_{32}, c_{33}\right) \end{pmatrix}$$

Angle Axis and Rotation Vector



- Angle axis parameterize the rotation by:

$$\chi_{R,AngleAxis} = \begin{pmatrix} \theta \\ \mathbf{n} \end{pmatrix}$$

Rotation angle θ
Rotation axis $\mathbf{n} \in \mathbb{R}^3$

- Rotation vector (aka Euler vectors)

$$\boldsymbol{\varphi} = \theta \cdot \mathbf{n} \in \mathbb{R}^3$$

Idea for the proof:
- Use unit-rotations to align z axis with n
- Rotate with angle θ around **n**
- Use unit-rotations to rotate back

[Robotics – Modelling, Planning and Control (Siciliano), p.53]

- Rotation matrix is given by:

$$\mathbf{C}_{\mathcal{AB}}(\theta,\mathbf{n}) = \cos(\theta)\mathbf{I}_{3\times 3} - \sin(\theta)[\mathbf{n}]_\times + (1-\cos(\theta))\mathbf{n}\mathbf{n}^T$$

$$\mathbf{C}_{\mathcal{AB}} = \begin{bmatrix} n_x^2(1-c_\theta)+c_\theta & n_x n_y(1-c_\theta)-n_z s_\theta & n_x n_z(1-c_\theta)+n_y s_\theta \\ n_x n_y(1-c_\theta)+n_z s_\theta & n_y^2(1-c_\theta)+c_\theta & n_y n_z(1-c_\theta)-n_x s_\theta \\ n_x n_z(1-c_\theta)-n_y s_\theta & n_y n_z(1-c_\theta)+n_x s_\theta & n_z^2(1-c_\theta)+c_\theta \end{bmatrix}$$

- Parameters from rotation matrix

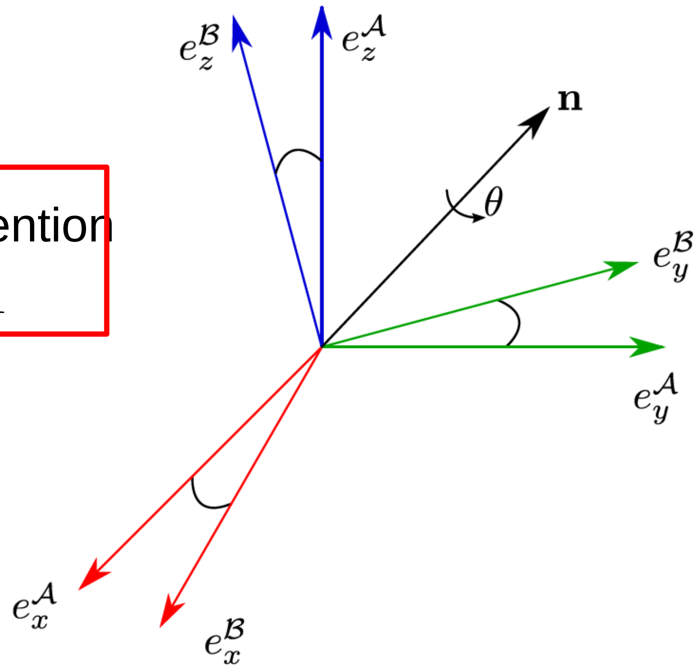
$$\theta = cos^{-1}\left(\frac{c_{11}+c_{22}+c_{33}-1}{2}\right)$$

$$\mathbf{n} = \frac{1}{2\boxed{sin(\theta)}}\begin{pmatrix} c_{32}-c_{23} \\ c_{13}-c_{31} \\ c_{21}-c_{12} \end{pmatrix}$$

Unit Quaternions
Rotation parameterization w/o singularity problem

Hamiltonian convention

$i^2 = j^2 = k^2 = ijk = -1$



- Complex numbers in 4D $\xi = \xi_0 + \xi_1 i + \xi_2 j + \xi_3 k$
- As vector $\chi_{R,quat} = \boldsymbol{\xi} = \begin{pmatrix} \xi_0 \\ \check{\boldsymbol{\xi}} \end{pmatrix} \in \mathbb{H}$
- Real part $\xi_0 = \cos\left(\frac{\|\boldsymbol{\varphi}\|}{2}\right) = \cos\left(\frac{\theta}{2}\right)$
- Imaginary part $\check{\boldsymbol{\xi}} = \sin\left(\frac{\|\boldsymbol{\varphi}\|}{2}\right)\frac{\boldsymbol{\varphi}}{\|\boldsymbol{\varphi}\|} = \sin\left(\frac{\theta}{2}\right)\mathbf{n} = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix}$
- Unitary constraint $\xi_0^2 + \xi_1^2 + \xi_2^2 + \xi_3^2 = 1$
- Inverse $\boldsymbol{\xi} = \begin{pmatrix} \xi \\ \check{\boldsymbol{\xi}} \end{pmatrix} \overset{inverse}{\longrightarrow} \boldsymbol{\xi}^{-1} = \begin{pmatrix} \xi \\ -\check{\boldsymbol{\xi}} \end{pmatrix}$
- Identity $\xi = \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix}^T$

Unit Quaternions ⇔ Rotation matrix

- Rotation matrix from unit quaternion

$$\mathbf{C}_{\mathcal{AD}} = \mathbb{I}_{3\times3} + 2\xi_0 \left[\check{\boldsymbol{\xi}}\right]_\times + 2\left[\check{\boldsymbol{\xi}}\right]_\times^2 = \left(2\xi_0^2 - 1\right)\mathbb{I}_{3\times3} + 2\xi_0\left[\check{\boldsymbol{\xi}}\right]_\times + 2\check{\boldsymbol{\xi}}\check{\boldsymbol{\xi}}^T$$
$$= \begin{bmatrix} \xi_0^2+\xi_1^2-\xi_2^2-\xi_3^2 & 2\xi_1\xi_2 - 2\xi_0\xi_3 & 2\xi_0\xi_2 + 2\xi_1\xi_3 \\ 2\xi_0\xi_3 + 2\xi_1\xi_2 & \xi_0^2-\xi_1^2+\xi_2^2-\xi_3^2 & 2\xi_2\xi_3 - 2\xi_0\xi_1 \\ 2\xi_1\xi_3 - 2\xi_0\xi_2 & 2\xi_0\xi_1 + 2\xi_2\xi_3 & \xi_0^2-\xi_1^2-\xi_2^2+\xi_3^2 \end{bmatrix}.$$

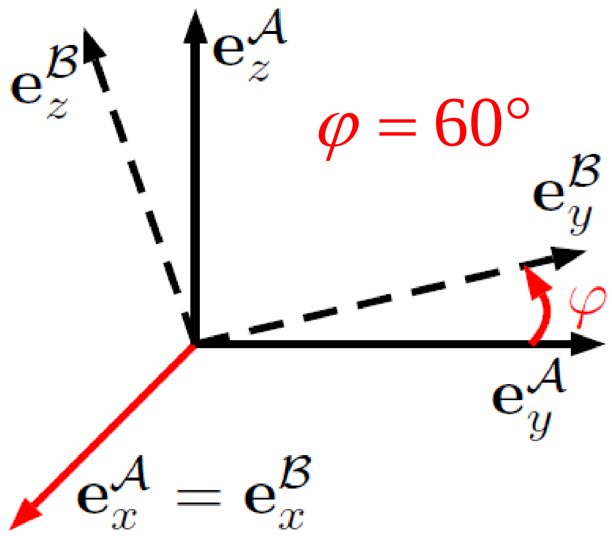
- Unit quaternions from rotation matrix

$$\boldsymbol{\chi}_{R,quat} = \boldsymbol{\xi}_{\mathcal{AD}} = \frac{1}{2}\begin{pmatrix} \sqrt{c_{11}+c_{22}+c_{33}+1} \\ sgn(c_{32}-c_{23})\sqrt{c_{11}-c_{22}-c_{33}+1} \\ sgn(c_{13}-c_{31})\sqrt{c_{22}-c_{33}-c_{11}+1} \\ sgn(c_{21}-c_{12})\sqrt{c_{33}-c_{11}-c_{22}+1} \end{pmatrix}$$

Quiz

- Rotation matrix

$$\mathbf{C}_{\mathcal{AB}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & -\sqrt{3}/2 \\ 0 & \sqrt{3}/2 & 1/2 \end{bmatrix}$$



- EulerZYX

$$\chi_{R,eulerZYX} = \begin{pmatrix} z \\ y \\ x \end{pmatrix} = \begin{pmatrix} atan2\left(c_{21}, c_{11}\right) \\ atan2\left(-c_{31}, \sqrt{c_{32}^2 + c_{33}^2}\right) \\ atan2\left(c_{32}, c_{33}\right) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \operatorname{atan2}(\sqrt{3}/2, 1/2) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 60° \end{pmatrix}$$

- Angle Axis

$$\theta = cos^{-1}\left(\frac{c_{11} + c_{22} + c_{33} - 1}{2}\right) = \cos^{-1}\left(\frac{1 + 1/2 + 1/2 - 1}{2}\right) = \cos^{-1}(1/2) = 60°$$

$$\mathbf{n} = \frac{1}{2sin(\theta)} \begin{pmatrix} c_{32} - c_{23} \\ c_{13} - c_{31} \\ c_{21} - c_{12} \end{pmatrix} = \frac{1}{2\sin(60°)} \begin{pmatrix} \sqrt{3}/2 - (-\sqrt{3}/2) \\ 0 - 0 \\ 0 - 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

- Quaternions

$$\chi_{R,quat} = \frac{1}{2} \begin{pmatrix} \sqrt{c_{11} + c_{22} + c_{33} + 1} \\ sgn(c_{32} - c_{23})\sqrt{c_{11} - c_{22} - c_{33} + 1} \\ sgn(c_{13} - c_{31})\sqrt{c_{22} - c_{33} - c_{11} + 1} \\ sgn(c_{21} - c_{12})\sqrt{c_{33} - c_{11} - c_{22} + 1} \end{pmatrix}$$

$$\xi_{\mathcal{AB}} = \frac{1}{2} \begin{pmatrix} \sqrt{1 + 1/2 + 1/2 + 1} \\ \sqrt{1 - 1/2 - 1/2 + 1} \\ \sqrt{1/2 - 1/2 - 1 + 1} \\ \sqrt{1/2 - 1 - 1/2 + 1} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \sqrt{3} \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\boxed{\begin{aligned} \zeta_0 &= \cos\left(\frac{\theta}{2}\right) \\ \check{\zeta} &= \sin\left(\frac{\theta}{2}\right)\mathbf{n} \end{aligned}}$$

Unit Quaternions
Algebra

- Product of quaternions $\xi_{\mathcal{AC}} = \xi_{\mathcal{AB}} \otimes \xi_{\mathcal{BC}} \longleftrightarrow \mathbf{C}_{\mathcal{AC}} = \mathbf{C}_{\mathcal{AB}} \cdot \mathbf{C}_{\mathcal{BC}}$
  - Given two quaternions **q** and **p**, the product is defined as

$$
\begin{aligned}
\boldsymbol{q} \otimes \boldsymbol{p} &= (q_0 + q_1\boldsymbol{i} + q_2\boldsymbol{j} + q_3\boldsymbol{k})(p_0 + p_1\boldsymbol{i} + p_2\boldsymbol{j} + p_3\boldsymbol{k}) \\
&= q_0p_0 + q_0p_1\boldsymbol{i} + q_0p_2\boldsymbol{j} + q_0p_3\boldsymbol{k} \\
&\quad + q_1p_0\boldsymbol{i} + q_1p_1\boldsymbol{ii} + q_1p_2\boldsymbol{ij} + q_1p_3\boldsymbol{ik} \\
&\quad + q_2p_0\boldsymbol{j} + q_2p_1\boldsymbol{ji} + q_2p_2\boldsymbol{jj} + q_2p_3\boldsymbol{jk} \\
&\quad + q_3p_0\boldsymbol{k} + q_3p_1\boldsymbol{ki} + q_3p_2\boldsymbol{kj} + q_3p_3\boldsymbol{kk} \\
&= q_0p_0 - q_1p_1 - q_2p_2 - q_3p_3 \\
&\quad + (q_0p_1 + q_1p_0 + q_2p_3 - q_3p_2)\boldsymbol{i} \\
&\quad + (q_0p_2 - q_1p_3 + q_2p_0 + q_3p_1)\boldsymbol{j} \\
&\quad + (q_0p_3 + q_1p_2 - q_2p_1 + q_3p_0)\boldsymbol{k} \\
&= \begin{bmatrix} q_0 & -q_1 & -q_2 & -q_3 \\ q_1 & q_0 & -q_3 & q_2 \\ q_2 & q_3 & q_0 & -q_1 \\ q_3 & -q_2 & q_1 & q_0 \end{bmatrix} \begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{pmatrix} = \underbrace{\begin{bmatrix} q_0 & -\check{\boldsymbol{q}}^T \\ \check{\boldsymbol{q}} & q_0\boldsymbol{I} + [\check{\boldsymbol{q}}]_\times \end{bmatrix}}_{=:\boldsymbol{M}_l(\boldsymbol{q})} \boldsymbol{p} = \boldsymbol{M}_l(\boldsymbol{q})\boldsymbol{p} \\
&= \begin{bmatrix} p_0 & -p_1 & -p_2 & -p_3 \\ p_1 & p_0 & p_3 & -p_2 \\ p_2 & -p_3 & p_0 & p_1 \\ p_3 & p_2 & -p_1 & p_0 \end{bmatrix} \begin{pmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{pmatrix} = \underbrace{\begin{bmatrix} p_0 & -\check{\boldsymbol{p}}^T \\ \check{\boldsymbol{p}} & p_0\boldsymbol{I} - [\check{\boldsymbol{p}}]_\times \end{bmatrix}}_{=:\boldsymbol{M}_r(\boldsymbol{p})} \boldsymbol{q} = \boldsymbol{M}_r(\boldsymbol{p})\boldsymbol{q}
\end{aligned}
$$

Hamiltonian convention

$$\xi = \xi_0 + \xi_1 i + \xi_2 j + \xi_3 k$$

$$i^2 = j^2 = k^2 = ijk = -1$$

$$ij = -ji = -ijk^2 = k$$

$$jk = -kj = i$$

$$ki = -ik = j$$

Unit Quaternions
Rotating a vector

- The pure (imaginary) quaternion of a coordinate vector ${}_I\boldsymbol{r}$ is given by

$$\boldsymbol{p}({}_I\boldsymbol{r}) = \begin{pmatrix} 0 \\ {}_I\boldsymbol{r} \end{pmatrix}$$

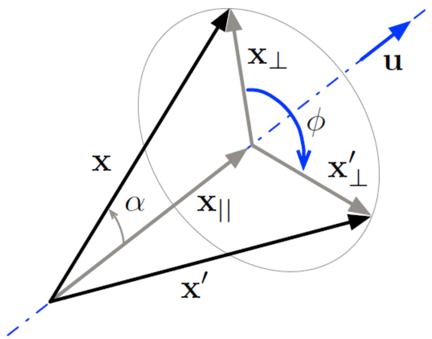
- Given the unit quaternion $\boldsymbol{\zeta}_{BI}$:

$$\boldsymbol{p}({}_B\boldsymbol{r}) = \boldsymbol{\zeta}_{BI} \otimes \boldsymbol{p}({}_I\boldsymbol{r}) \otimes \boldsymbol{\zeta}_{BI}^T \qquad \longleftrightarrow \qquad {}_{\mathcal{B}}\mathbf{r} = \mathbf{C}_{\mathcal{B}\mathcal{I}} \cdot {}_{\mathcal{I}}\mathbf{r}$$

- Proof (see Quaternion Kinematics by Joan Solà on lecture homepage)
  - Decompose vector in parallel and orthogonal part to get vector rotation formula

$$\boxed{\mathbf{x}' = \mathbf{x}_{||} + \mathbf{x}_{\perp} \cos\phi + (\mathbf{u} \times \mathbf{x}) \sin\phi}$$

  - Show that equation above does exactly the same



$$\mathbf{x} = \mathbf{x}_{||} + \mathbf{x}_{\perp}$$
$$\mathbf{x}_{||} = \mathbf{u}\,\mathbf{u}^\top \mathbf{x}$$
$$\mathbf{x}_{\perp} = \mathbf{x} - \mathbf{u}\,\mathbf{u}^\top \mathbf{x}$$

Unit Quaternion

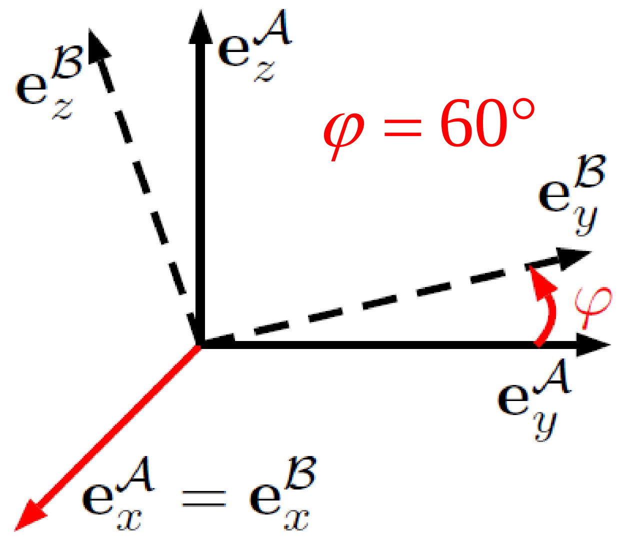
Derivation of rotation matrix

- Derivation of rotation matrix ($\boldsymbol{\zeta} = \boldsymbol{\zeta}_{BI}$):
  - $\boldsymbol{p}({}_B\boldsymbol{r}) = \boldsymbol{\zeta} \otimes \boldsymbol{p}({}_I\boldsymbol{r}) \otimes \boldsymbol{\zeta}^T = \boldsymbol{M}_l(\boldsymbol{\zeta})\boldsymbol{M}_r(\boldsymbol{\zeta}^T)\begin{pmatrix} 0 \\ {}_I\boldsymbol{r} \end{pmatrix}$
  - $\begin{pmatrix} 0 \\ {}_B\boldsymbol{r} \end{pmatrix} = \begin{bmatrix} \zeta_0 & -\check{\boldsymbol{\zeta}}^T \\ \check{\boldsymbol{\zeta}} & \zeta_0\boldsymbol{I} + [\check{\zeta}]_\times \end{bmatrix}\begin{bmatrix} \zeta_0 & \check{\boldsymbol{\zeta}}^T \\ -\check{\boldsymbol{\zeta}} & \zeta_0\boldsymbol{I} + [\check{\zeta}]_\times \end{bmatrix}\begin{pmatrix} 0 \\ {}_I\boldsymbol{r} \end{pmatrix}$
  - $\begin{pmatrix} 0 \\ {}_B\boldsymbol{r} \end{pmatrix} = \begin{bmatrix} \zeta_0^2 + |\check{\zeta}|^2 & \zeta_0\check{\boldsymbol{\zeta}}^T - \zeta_0\check{\boldsymbol{\zeta}}^T - \check{\boldsymbol{\zeta}}^T[\check{\zeta}]_\times \\ \zeta_0\check{\boldsymbol{\zeta}} - \zeta_0\check{\boldsymbol{\zeta}} - [\check{\zeta}]_\times\check{\boldsymbol{\zeta}} & \check{\boldsymbol{\zeta}}\check{\boldsymbol{\zeta}}^T + \zeta_0^2\boldsymbol{I} + 2\zeta_0[\check{\zeta}]_\times + \underline{[\check{\zeta}]_\times[\check{\zeta}]_\times} \end{bmatrix}\begin{pmatrix} 0 \\ {}_I\boldsymbol{r} \end{pmatrix}$

    $\boxed{[\check{\zeta}]_\times[\check{\zeta}]_\times = \check{\zeta}\check{\zeta}^T - |\check{\zeta}|^2\mathbf{I}}$
  - $\begin{pmatrix} 0 \\ {}_B\boldsymbol{r} \end{pmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \left(\zeta_0^2 - |\check{\zeta}|^2\right)\boldsymbol{I} + 2\zeta_0[\check{\zeta}]_\times + \boldsymbol{2}\check{\boldsymbol{\zeta}}\check{\boldsymbol{\zeta}}^T \end{bmatrix}\begin{pmatrix} 0 \\ {}_I\boldsymbol{r} \end{pmatrix}$
  - ${}_B\boldsymbol{r} = \left(\left(\zeta_0^2 - |\check{\zeta}|^2\right)\boldsymbol{I} + 2\zeta_0[\check{\zeta}]_\times + \boldsymbol{2}\check{\boldsymbol{\zeta}}\check{\boldsymbol{\zeta}}^T\right)\, {}_I\boldsymbol{r}$
  - $\boldsymbol{C}(\boldsymbol{\zeta}) = (2\zeta_0^2 - 1)\boldsymbol{I} + 2\zeta_0[\check{\zeta}]_\times + \boldsymbol{2}\check{\boldsymbol{\zeta}}\check{\boldsymbol{\zeta}}^T$

$$\boxed{\begin{aligned} \boldsymbol{M}_r(\boldsymbol{\zeta}) &= \begin{bmatrix} \zeta_0 & -\check{\boldsymbol{\zeta}}^T \\ \check{\boldsymbol{\zeta}} & \zeta_0\boldsymbol{I} - [\check{\zeta}]_\times \end{bmatrix} \\ [\check{\boldsymbol{\zeta}}^T]_\times &= -[\check{\zeta}]_\times \\ \boldsymbol{\zeta}^{-1} = \boldsymbol{\zeta}^T &= \begin{pmatrix} \zeta_0 \\ -\check{\boldsymbol{\zeta}} \end{pmatrix} \end{aligned}}$$

Quiz 2



- Given a vector in $\mathcal{A}$ frame $\quad {}_{\mathcal{A}}\mathbf{r} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

- Rotate this to $\mathcal{B}$ frame using quaternions

$$\xi_{\mathcal{AB}} = \frac{1}{2}\begin{pmatrix} \sqrt{3} \\ 1 \\ 0 \\ 0 \end{pmatrix} \qquad \xi_{\mathcal{BA}} = \frac{1}{2}\begin{pmatrix} \sqrt{3} \\ -1 \\ 0 \\ 0 \end{pmatrix}$$

$$\mathbf{p}({}_{\mathcal{B}}\mathbf{r}) = \begin{pmatrix} 0 \\ {}_{\mathcal{B}}\mathbf{r} \end{pmatrix} = \xi_{\mathcal{BA}} \otimes \mathbf{p}({}_{\mathcal{A}}\mathbf{r}) \otimes \xi_{\mathcal{BA}}^T = \mathbf{M}_l(\xi_{\mathcal{BA}})\mathbf{M}_r(\xi_{\mathcal{BA}}^T)\begin{pmatrix} 0 \\ {}_{\mathcal{A}}\mathbf{r} \end{pmatrix} = \frac{1}{2}\begin{bmatrix} \sqrt{3} & 1 & 0 & 0 \\ -1 & \sqrt{3} & 0 & 0 \\ 0 & 0 & \sqrt{3} & 1 \\ 0 & 0 & -1 & \sqrt{3} \end{bmatrix} \frac{1}{2}\begin{bmatrix} \sqrt{3} & -1 & 0 & 0 \\ 1 & \sqrt{3} & 0 & 0 \\ 0 & 0 & \sqrt{3} & 1 \\ 0 & 0 & -1 & \sqrt{3} \end{bmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1/2 \\ -\sqrt{3}/2 \end{pmatrix}$$

Direct calculation using complex numbers

$$\xi_{\mathcal{BA}} \otimes \mathbf{p}({}_{\mathcal{A}}\mathbf{r}) \otimes \xi_{\mathcal{BA}}^T => \frac{1}{2}\left(\sqrt{3}-i\right)\cdot(j)\cdot\frac{1}{2}\left(\sqrt{3}+i\right) = \frac{1}{2}\frac{1}{2}\left(\sqrt{3}j\sqrt{3}+\sqrt{3}ji-ij\sqrt{3}-iji\right) = \frac{1}{2}\frac{1}{2}\left(3j+\sqrt{3}ji+\sqrt{3}ji+iij\right) = \frac{1}{2}\frac{1}{2}\left(3j+\sqrt{3}ji+\sqrt{3}ji+iij\right)\frac{1}{2}\frac{1}{2}\left(3j-\sqrt{3}k-\sqrt{3}k-j\right) = \frac{1}{2}\frac{1}{2}\left(2j-2\sqrt{3}k\right)$$

Unit Quaternion
Link to angle axis

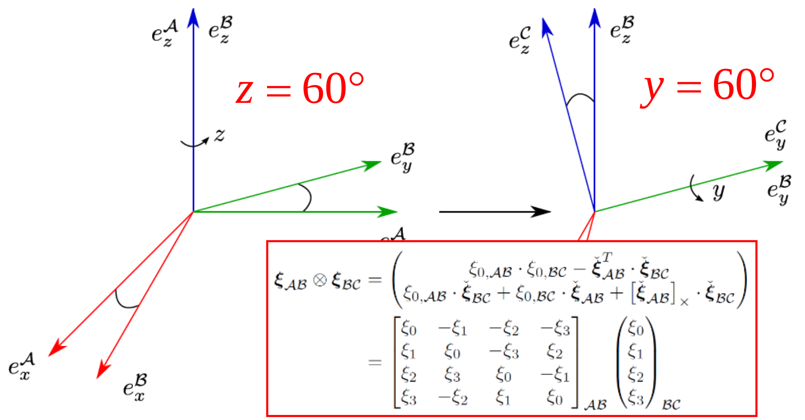
- Given the rotation matrix $\boldsymbol{C}(\boldsymbol{\zeta}) = (2\zeta_0^2 - 1)\boldsymbol{I} + 2\zeta_0\left[\check{\boldsymbol{\zeta}}\right]_\times + \mathbf{2}\check{\boldsymbol{\zeta}}\check{\boldsymbol{\zeta}}^T$

- Use this with the angle axis representation
  - $\zeta_0 = \cos\left(\frac{\theta}{2}\right)$
  - $\check{\boldsymbol{\zeta}} = \sin\left(\frac{\theta}{2}\right)\mathbf{n}$

➢ $$\boldsymbol{C}(\boldsymbol{\zeta}(\theta, \mathbf{n})) = \left(2\cos^2\left(\frac{\theta}{2}\right) - 1\right)\boldsymbol{I} + 2\cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)[\mathbf{n}]_\times + \mathbf{2}\sin^2\left(\frac{\theta}{2}\right)\mathbf{n}\mathbf{n}^T$$
$$= \cos(\theta)\,\boldsymbol{I} + \sin(\theta)\,[\mathbf{n}]_\times + (\mathbf{1} - \cos(\theta))\mathbf{n}\mathbf{n}^T$$

$\boxed{\text{Rodrigues' Formula}}$

Quiz 3



$$\boldsymbol{\xi}_{\mathcal{AB}} \otimes \boldsymbol{\xi}_{\mathcal{BC}} = \begin{pmatrix} \xi_{0,\mathcal{AB}} \cdot \xi_{0,\mathcal{BC}} - \check{\boldsymbol{\xi}}_{\mathcal{AB}}^T \cdot \check{\boldsymbol{\xi}}_{\mathcal{BC}} \\ \xi_{0,\mathcal{AB}} \cdot \check{\boldsymbol{\xi}}_{\mathcal{BC}} + \xi_{0,\mathcal{BC}} \cdot \check{\boldsymbol{\xi}}_{\mathcal{AB}} + [\check{\boldsymbol{\xi}}_{\mathcal{AB}}]_\times \cdot \check{\boldsymbol{\xi}}_{\mathcal{BC}} \end{pmatrix} = \begin{bmatrix} \xi_0 & -\xi_1 & -\xi_2 & -\xi_3 \\ \xi_1 & \xi_0 & -\xi_3 & \xi_2 \\ \xi_2 & \xi_3 & \xi_0 & -\xi_1 \\ \xi_3 & -\xi_2 & \xi_1 & \xi_0 \end{bmatrix}_{\mathcal{AB}} \begin{pmatrix} \xi_0 \\ \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix}_{\mathcal{BC}}$$

- Rotation matrix $\mathbf{C}_{\mathcal{AC}} = ?$ $\quad \mathbf{C}_{\mathcal{AB}} = \begin{bmatrix} 1/2 & -\sqrt{3}/2 & 0 \\ \sqrt{3}/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\quad \mathbf{C}_{\mathcal{BC}} = \begin{bmatrix} 1/2 & 0 & \sqrt{3}/2 \\ 0 & 1 & 0 \\ -\sqrt{3}/2 & 0 & 1/2 \end{bmatrix}$

- Quaternion $\xi_{\mathcal{AB}} = \frac{1}{2}\begin{pmatrix} \sqrt{3} \\ 0 \\ 0 \\ 1 \end{pmatrix}$ $\quad \xi_{\mathcal{BC}} = \frac{1}{2}\begin{pmatrix} \sqrt{3} \\ 0 \\ 1 \\ 0 \end{pmatrix}$

$$\xi_{\mathcal{AC}} = \xi_{\mathcal{AB}} \otimes \xi_{\mathcal{BC}} = \frac{1}{2}\begin{pmatrix} \sqrt{3} \\ 0 \\ 0 \\ 1 \end{pmatrix} \otimes \frac{1}{2}\begin{pmatrix} \sqrt{3} \\ 0 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{2}\frac{1}{2}\begin{bmatrix} \sqrt{3} & 0 & 0 & -1 \\ 0 & \sqrt{3} & -1 & 0 \\ 0 & 1 & \sqrt{3} & 0 \\ 1 & 0 & 0 & \sqrt{3} \end{bmatrix}\begin{pmatrix} \sqrt{3} \\ 0 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{2}\frac{1}{2}\begin{pmatrix} 3 \\ -1 \\ \sqrt{3} \\ \sqrt{3} \end{pmatrix}$$

$$\begin{aligned} \mathbf{C} &= \mathbb{I}_{3\times3} + 2\xi_0 \left[\check{\boldsymbol{\xi}}\right]_\times + 2\left[\check{\boldsymbol{\xi}}\right]_\times^2 = \left(2\xi_0^2 - 1\right)\mathbb{I}_{3\times3} + 2\xi_0\left[\check{\boldsymbol{\xi}}\right]_\times + 2\check{\boldsymbol{\xi}}\check{\boldsymbol{\xi}}^T \\ &= \begin{bmatrix} \xi_0^2 + \xi_1^2 - \xi_2^2 - \xi_3^2 & 2\xi_1\xi_2 - 2\xi_0\xi_3 & 2\xi_0\xi_2 + 2\xi_1\xi_3 \\ 2\xi_0\xi_3 + 2\xi_1\xi_2 & \xi_0^2 - \xi_1^2 + \xi_2^2 - \xi_3^2 & 2\xi_2\xi_3 - 2\xi_0\xi_1 \\ 2\xi_1\xi_3 - 2\xi_0\xi_2 & 2\xi_0\xi_1 + 2\xi_2\xi_3 & \xi_0^2 - \xi_1^2 - \xi_2^2 + \xi_3^2 \end{bmatrix}. \end{aligned}$$

$\Rightarrow$

$$\mathbf{C}_{\mathcal{AC}} = \left(\frac{1}{2}\frac{1}{2}\right)^2 \begin{bmatrix} 9+1-3-3 & -2\sqrt{3}-6\sqrt{3} & 6\sqrt{3}-2\sqrt{3} \\ 6\sqrt{3}-2\sqrt{3} & 9-1+3-3 & 2\sqrt{3}\sqrt{3}+6 \\ -2\sqrt{3}-6\sqrt{3} & -6+2\sqrt{3}\sqrt{3} & 9-1-3+3 \end{bmatrix}$$

Time Derivatives and Rotational Velocity

- What is the relation $\boldsymbol{\omega}_{\mathcal{AD}} \Leftrightarrow \dot{\boldsymbol{\chi}}_{\mathcal{AD}}$
- Analog to linear velocity: Find $\mathbf{E}_R(\boldsymbol{\chi}_R)$, s.t. $_{\mathcal{A}}\boldsymbol{\omega}_{\mathcal{AB}} = \mathbf{E}_R(\boldsymbol{\chi}_R) \cdot \dot{\boldsymbol{\chi}}_R$

Time Derivatives and Rotational Velocity

ZYX example

$$ {}_{\mathcal{A}}\boldsymbol{\omega}_{\mathcal{AD}} = {}_{\mathcal{A}}\boldsymbol{\omega}_{\mathcal{AB}} + {}_{\mathcal{A}}\boldsymbol{\omega}_{\mathcal{BC}} + {}_{\mathcal{A}}\boldsymbol{\omega}_{\mathcal{CD}} $$

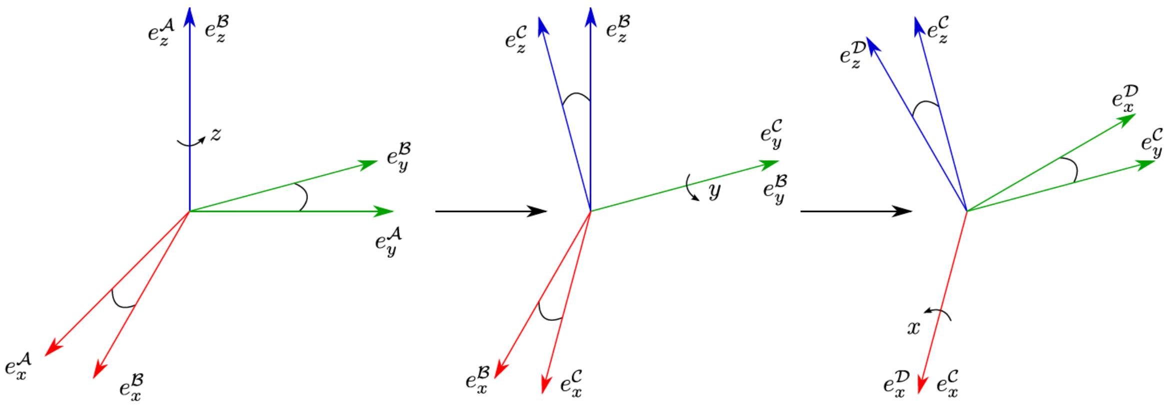
$$ = {}_{\mathcal{A}}\boldsymbol{\omega}_{\mathcal{AB}} + \mathbf{C}_{\mathcal{AB}} \cdot {}_{\mathcal{B}}\boldsymbol{\omega}_{\mathcal{BC}} + \mathbf{C}_{\mathcal{AB}} \cdot \mathbf{C}_{\mathcal{BC}} \cdot {}_{\mathcal{C}}\boldsymbol{\omega}_{\mathcal{CD}} $$

$$ = {}_{\mathcal{A}}\mathbf{e}_z^{\mathcal{A}} \cdot \dot{z} + \mathbf{C}_{\mathcal{AB}} \cdot {}_{\mathcal{B}}\mathbf{e}_y^{\mathcal{B}} \cdot \dot{y} + \mathbf{C}_{\mathcal{AB}} \cdot \mathbf{C}_{\mathcal{BC}} \cdot {}_{\mathcal{C}}\mathbf{e}_x^{\mathcal{C}} \cdot \dot{x} $$

$$ = \begin{bmatrix} {}_{\mathcal{A}}\mathbf{e}_z^{\mathcal{A}} & \mathbf{C}_{\mathcal{AB}} \cdot {}_{\mathcal{B}}\mathbf{e}_y^{\mathcal{B}} & \mathbf{C}_{\mathcal{AB}} \cdot \mathbf{C}_{\mathcal{BC}} \cdot {}_{\mathcal{C}}\mathbf{e}_x^{\mathcal{C}} \end{bmatrix} \begin{pmatrix} \dot{z} \\ \dot{y} \\ \dot{x} \end{pmatrix} $$

$$ \mathbf{C}_{\mathcal{AB}} \cdot {}_{\mathcal{B}}\mathbf{e}_y^{\mathcal{B}} = \begin{bmatrix} c_z & -s_z & 0 \\ s_z & c_z & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -s_z \\ c_z \\ 0 \end{pmatrix} $$

$$ \mathbf{C}_{\mathcal{AB}} \cdot \mathbf{C}_{\mathcal{BC}} \cdot {}_{\mathcal{C}}\mathbf{e}_x^{\mathcal{C}} = \begin{bmatrix} c_z & -s_z & 0 \\ s_z & c_z & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_y & 0 & s_y \\ 0 & 1 & 0 \\ -s_y & 0 & c_y \end{bmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} c_y c_z \\ c_y s_z \\ -s_y \end{pmatrix} $$



$$ \boldsymbol{\omega} = \begin{bmatrix} 0 & -s_z & c_y c_z \\ 0 & c_z & c_y s_z \\ 1 & 0 & -s_y \end{bmatrix} \dot{\boldsymbol{\chi}} $$

$$ \det\left(\mathbf{E}_{R,eulerZYX}\right) = -\cos(y) $$

$$ \mathbf{E}_{R,eulerZYX}^{-1} = \begin{bmatrix} \frac{\cos(z)\sin(y)}{\cos(y)} & \frac{\sin(y)\sin(z)}{\cos(y)} & 1 \\ -\sin(z) & \cos(z) & 0 \\ \frac{\cos(z)}{\cos(y)} & \frac{\sin(z)}{\cos(y)} & 0 \end{bmatrix} $$

Derivative Angle Axis, Rotation Vector, Quaternions ⇔ Angular Velocity

$$\boxed{{}_{\mathcal{A}}\boldsymbol{\omega}_{\mathcal{AB}} = \mathbf{E}_R(\boldsymbol{\chi}_R) \cdot \dot{\boldsymbol{\chi}}_R}$$

- Angle Axis

$$\mathbf{E}_{R,angleaxis} = \begin{bmatrix} \mathbf{n} & \sin\theta \mathbb{I}_{3\times3} + (1-\cos\theta)[\mathbf{n}]_\times \end{bmatrix}$$

$$\mathbf{E}^{-1}_{R,angleaxis} = \begin{bmatrix} \mathbf{n}^T \\ -\frac{1}{2}\frac{\sin\theta}{1-cos\theta}[\mathbf{n}]^2_\times - \frac{1}{2}[\mathbf{n}]_\times \end{bmatrix}$$

- Rotation Vector

$$\mathbf{E}_{R,rotationvector} = \left[\mathbb{I}_{3\times3} + [\boldsymbol{\varphi}]_\times \left(\frac{1-\cos\|\boldsymbol{\varphi}\|}{\|\boldsymbol{\varphi}\|^2}\right) + [\boldsymbol{\varphi}]^2_\times \left(\frac{\|\boldsymbol{\varphi}\| - \sin\|\boldsymbol{\varphi}\|}{\|\boldsymbol{\varphi}\|^3}\right)\right]$$

$$\mathbf{E}^{-1}_{R,rotationvector} = \left[\mathbb{I}_{3\times3} - \frac{1}{2}[\boldsymbol{\varphi}]_\times + [\boldsymbol{\varphi}]^2_\times \frac{1}{\|\boldsymbol{\varphi}\|^2}\left(1 - \frac{\|\boldsymbol{\varphi}\|}{2}\frac{\sin\|\boldsymbol{\varphi}\|}{1-\cos\|\boldsymbol{\varphi}\|}\right)\right]$$

- Quaternion

$$\mathbf{E}_{R,quat} = 2\mathbf{H}(\boldsymbol{\xi}),$$

$$\mathbf{E}^{-1}_{R,quat} = \frac{1}{2}\mathbf{H}(\boldsymbol{\xi})^T$$

with

$$\mathbf{H}(\boldsymbol{\xi}) = \begin{bmatrix} -\check{\boldsymbol{\xi}} & [\check{\boldsymbol{\xi}}]_\times + \xi_0 \mathbb{I}_{3\times3} \end{bmatrix} \in \mathbb{R}^{3\times4}$$

$$= \begin{bmatrix} -\xi_1 & \xi_0 & -\xi_3 & \xi_2 \\ -\xi_2 & \xi_3 & \xi_0 & -\xi_1 \\ -\xi_3 & -\xi_2 & \xi_1 & \xi_0 \end{bmatrix}.$$

Position and Orientation of a Single Body

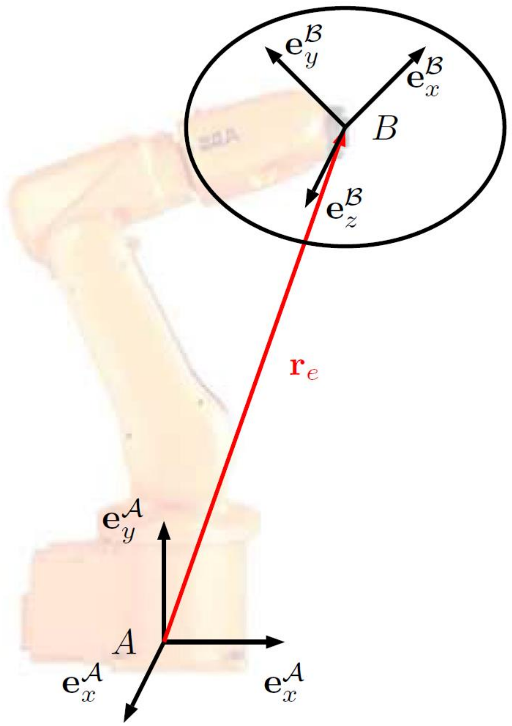
- Position vector: $\mathbf{r}_e = \mathbf{r}_e(\chi) \in \mathbb{R}^3$
  - Cartesian $\chi_{Pc} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$
  - Cylindrical coordinates $\chi_{Pz} = \begin{pmatrix} \rho \\ \theta \\ z \end{pmatrix}$
  - Spherical coordinates $\chi_{Ps} = \begin{pmatrix} r \\ \theta \\ \phi \end{pmatrix}$
- Rotation $\phi_e = \phi_e(\chi_R) \in SO(3)$

$$\boxed{\mathbf{x}_e = \begin{pmatrix} \mathbf{r}_e \\ \boldsymbol{\phi}_e \end{pmatrix} \in SE(3)}$$

Rotation matrix: $\mathbf{C}_{\mathcal{AB}} = \begin{bmatrix} {}_{\mathcal{A}}\mathbf{e}_x^{\mathcal{B}} & {}_{\mathcal{A}}\mathbf{e}_y^{\mathcal{B}} & {}_{\mathcal{A}}\mathbf{e}_z^{\mathcal{B}} \end{bmatrix}$

Euler Angles $\chi_{R,eulerZYX} = \begin{pmatrix} z \\ y \\ x \end{pmatrix}$

Quaternions $\chi_{R,quat} = \boldsymbol{\xi} = \begin{pmatrix} \xi_0 \\ \check{\boldsymbol{\xi}} \end{pmatrix}$



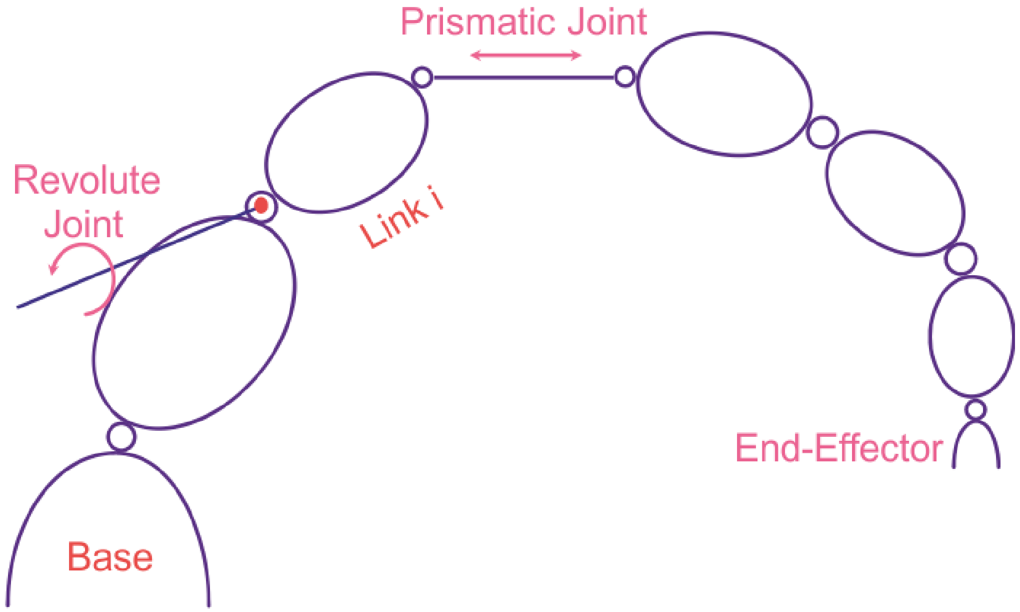
Mini-Introduction to Multi-body Kinematics

151-0851-00 V

lecture:	CAB G11	Tuesday 10:15 – 12:00, every week
exercise:	HG E1.2	Wednesday 8:15 – 10:00, according to schedule (about every 2nd week)

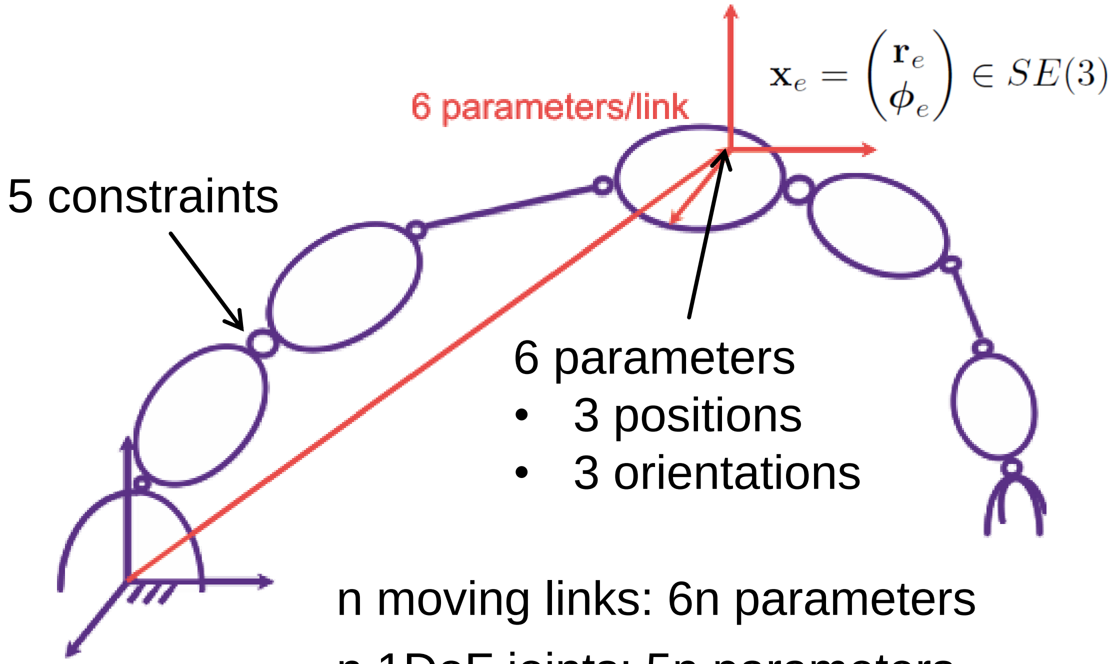
Marco Hutter, Roland Siegwart, and Thomas Stastny

Classical Serial Kinematic Linkages
Generalized robot arm



- n_j joints
  - revolute (1DOF)
  - prismatic (1DOF)
- $n_l = n_j + 1$ links
  - n_j moving links
  - 1 fixed link

Configuration Parameters
Generalized coordinates



n moving links: 6n parameters

n 1DoF joints: 5n parameters

6n – 5n = n DoFs

Generalized coordinates

A set of scalar parameters **q** that describe the robot's configuration

- Must be **complete**
- (Must be **independent**) => minimal coordinates
- Is **not unique**

$$\mathbf{q} = \begin{pmatrix} q_1 \\ \vdots \\ q_{n_j} \end{pmatrix} \in \mathbb{R}^{n_j}$$

Degrees of Freedom

- Nr of minimal coordinates

Forward Kinematics

- End-effector configuration as a function of generalized coordinates

$$\chi_e = \chi_e(\mathbf{q}) \in \mathbb{R}^{n_e}$$

- For multi-body system, use transformation matrices

$$\mathbf{T}_{\mathcal{I}\mathcal{E}}(\mathbf{q}) = \mathbf{T}_{\mathcal{I}0} \cdot \left(\prod_{k=1}^{n_j} \mathbf{T}_{k-1,k}(q_k) \right) \cdot \mathbf{T}_{n_j\mathcal{E}} = \begin{bmatrix} \mathbf{C}_{\mathcal{I}\mathcal{E}}(\mathbf{q}) & {}_{\mathcal{I}}\mathbf{r}_{IE}(\mathbf{q}) \\ \mathbf{0}_{1\times 3} & 1 \end{bmatrix}$$

- Note: depending on the selected end-effector parameterization, it is not possible to analytically write down end-effector parameters!

Forward Kinematics
Simple example

$$\mathbf{q} = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} = \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{pmatrix}$$

- What is the end-effector configuration as a function of generalized coordinates?

$$\mathbf{T}_{IE} = \mathbf{T}_{I0} \cdot \mathbf{T}_{01} \cdot \mathbf{T}_{12} \cdot \mathbf{T}_{23} \cdot \mathbf{T}_{3E}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 & 0 & s_1 & 0 \\ 0 & 1 & 0 & 0 \\ -s_1 & 0 & c_1 & l_0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_2 & 0 & s_2 & 0 \\ 0 & 1 & 0 & 0 \\ -s_2 & 0 & c_2 & l_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_3 & 0 & s_3 & 0 \\ 0 & 1 & 0 & 0 \\ -s_3 & 0 & c_3 & l_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & l_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \ldots = \begin{bmatrix} c_{123} & 0 & s_{123} & l_1 s_1 + l_2 s_{12} + l_3 s_{123} \\ 0 & 1 & 0 & 0 \\ -s_{123} & 0 & c_{123} & l_0 + l_1 c_1 + l_2 c_{12} + l_3 c_{123} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$