# Lecture «Robot Dynamics»: Multi-body Kinematics

**151-0851-00 V**

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|---|---|---|
| lecture: | CAB G11 | Tuesday 10:15 – 12:00, every week |
| exercise: | HG E1.2 | Wednesday 8:15 – 10:00, according to schedule (about every 2nd week) |

Marco Hutter, Roland Siegwart, and Thomas Stastny

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|---|---|---|---|---|---|
| 19.09.2017 | Intro and Outline | Course Introduction; Recapitulation Position, Linear Velocity | | | |
| 26.09.2017 | Kinematics 1 | Rotation and Angular Velocity; Rigid Body Formulation, Transformation | 26.09.2017 | Exercise 1a | Kinematics Modeling the ABB arm |
| 03.10.2017 | Kinematics 2 | Kinematics of Systems of Bodies; Jacobians | 03.10.2017 | Exercise 1b | Differential Kinematics of the ABB arm |
| 10.10.2017 | Kinematics 3 | Kinematic Control Methods: Inverse Differential Kinematics, Inverse Kinematics; Rotation Error; Multi-task Control | 10.10.2017 | Exercise 1c | Kinematic Control of the ABB Arm |
| 17.10.2017 | Dynamics L1 | Multi-body Dynamics | 17.10.2017 | Exercise 2a | Dynamic Modeling of the ABB Arm |
| 24.10.2017 | Dynamics L2 | Floating Base Dynamics | 24.10.2017 | | |
| 31.10.2017 | Dynamics L3 | Dynamic Model Based Control Methods | 31.10.2017 | Exercise 2b | Dynamic Control Methods Applied to the ABB arm |
| 07.11.2017 | Legged Robot | Dynamic Modeling of Legged Robots & Control | 07.11.2017 | Exercise 3 | Legged robot |
| 14.11.2017 | Case Studies 1 | Legged Robotics Case Study | 14.11.2017 | | |
| 21.11.2017 | Rotorcraft | Dynamic Modeling of Rotorcraft & Control | 21.11.2017 | Exercise 4 | Modeling and Control of Multicopter |
| 28.11.2017 | Case Studies 2 | Rotor Craft Case Study | 28.11.2017 | | |
| 05.12.2017 | Fixed-wing | Dynamic Modeling of Fixed-wing & Control | 05.12.2017 | Exercise 5 | Fixed-wing Control and Simulation |
| 12.12.2017 | Case Studies 3 | Fixed-wing Case Study (Solar-powered UAVs - AtlantikSolar, Vertical Take-off and Landing UAVs – Wingtra) | | | |
| 19.12.2017 | Summery and Outlook | Summery; Wrap-up; Exam | | | |

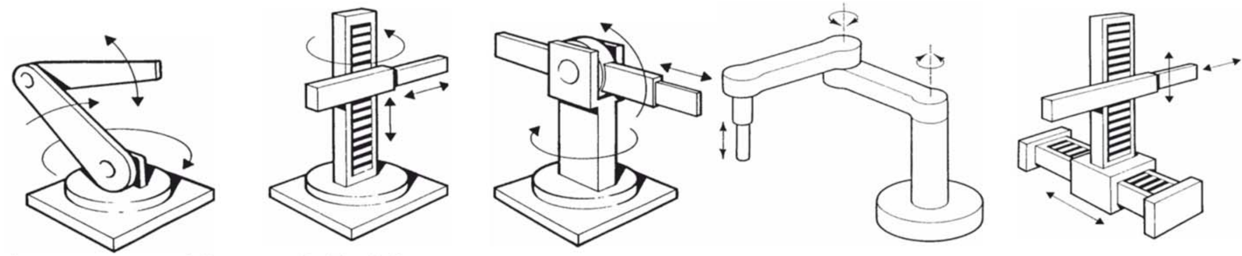
# Multi-body Kinematics
Intro

- Machines are built and controlled to
  - achieve extremely accurate positions,
  - independent of the load the robot carries

  ➢ Very stiff structure
  ➢ Play-free gears and transmissions
  ➢ High-accurate joint sensors

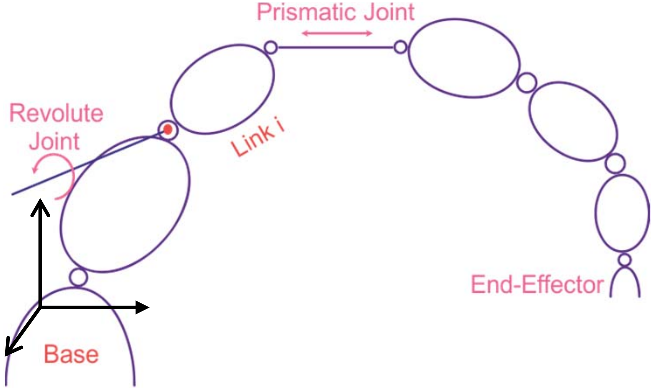
  ➢ End-effector accuracy +/- 0.02mm!

- Large variety of robot arms

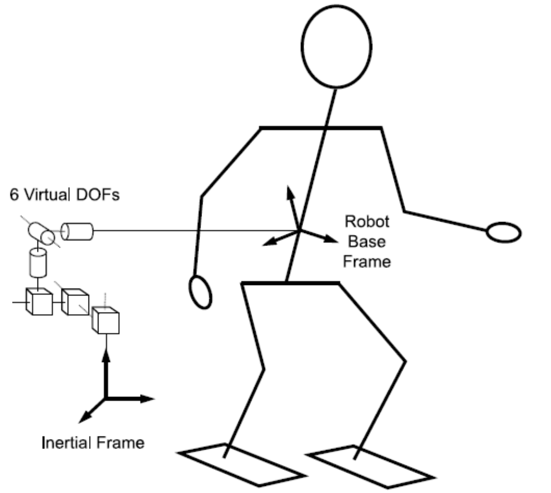



# Fixed Base vs. Floating Base Robot

- Base frame is rigidly connected to ground
  - Often indicated as CS 0

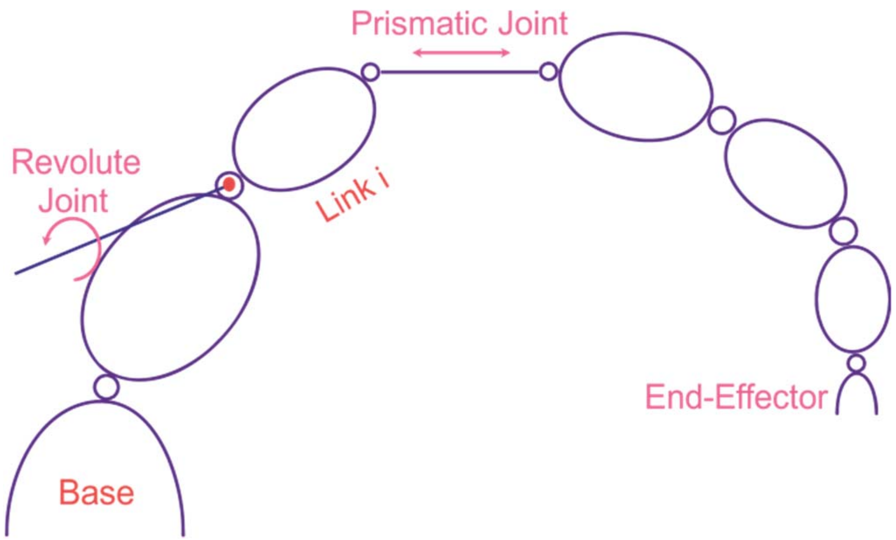



- Base frame is free floating
  - Often indicated as CS $\mathcal{B}$ (base)
  - 6 unactuated DOFs!



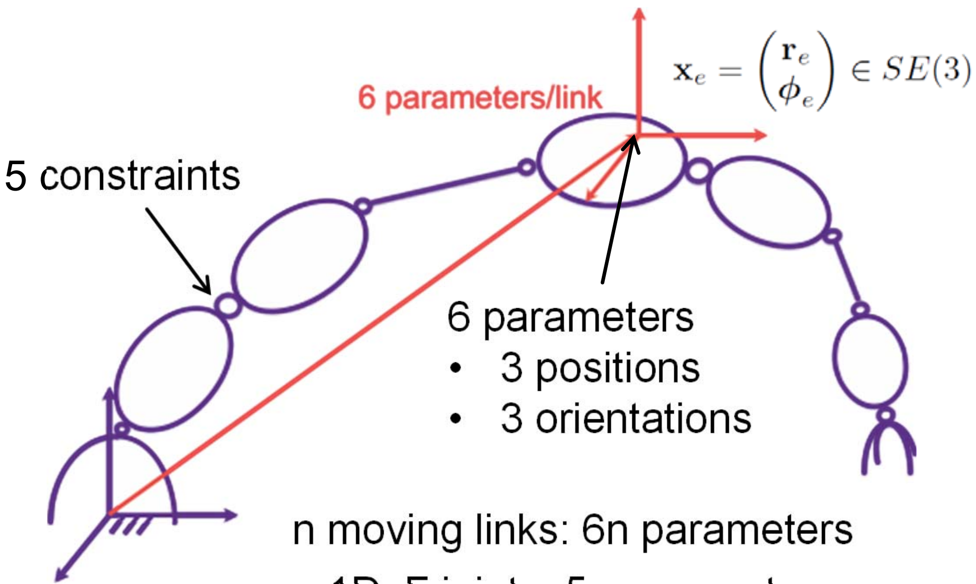
# Classical Serial Kinematic Linkages

## Generalized robot arm




- $n_j$ joints
  - revolute (1DOF)
  - prismatic (1DOF)
- $n_l = n_j + 1$ links
  - $n_j$ moving links
  - 1 fixed link

# Configuration Parameters
## Generalized coordinates




n moving links: 6n parameters

n 1DoF joints: 5n parameters

6n – 5n = n DoFs

### Generalized coordinates

A set of scalar parameters **q** that describe the robot's configuration

- Must be **complete**
- (Must be **independent**)
  => minimal coordinates
- Is **not unique**

$$\mathbf{q} = \begin{pmatrix} q_1 \\ \vdots \\ q_{n_j} \end{pmatrix} \in \mathbb{R}^{n_j}$$

### Degrees of Freedom

- Nr of minimal coordinates

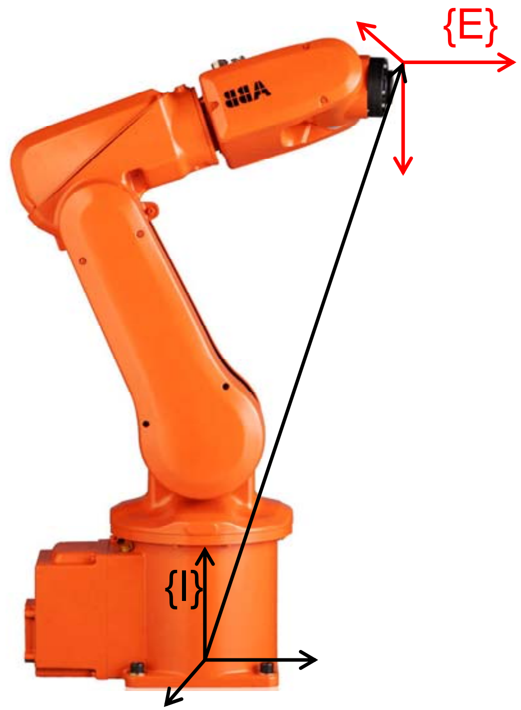
# End-effector Configuration Parameters

- End-effector configuration parameters
  - A set of m parameters that completely specify the end-effector position and orientation with respect to $\mathcal{I}$

$$\boldsymbol{\chi}_e = \begin{pmatrix} \boldsymbol{\chi}_{eP} \\ \boldsymbol{\chi}_{eR} \end{pmatrix} = \begin{pmatrix} \chi_1 \\ \vdots \\ \chi_m \end{pmatrix} \in \mathbb{R}^m$$

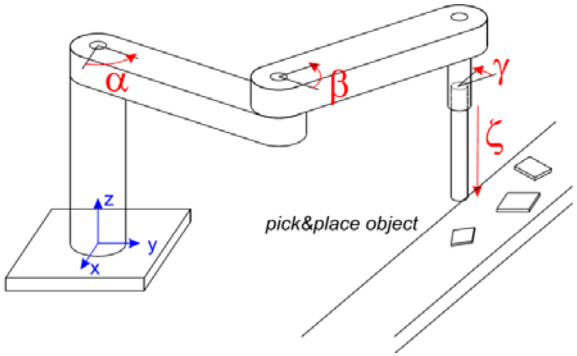
- Operational space coordinates
  - the $m_0$ configuration parameters are independent => $m_0$ number of degrees of freedom of end-effector

$$\boldsymbol{\chi}_o = \begin{pmatrix} \boldsymbol{\chi}_{oP} \\ \boldsymbol{\chi}_{oR} \end{pmatrix} = \begin{pmatrix} \chi_1 \\ \vdots \\ \chi_{m_0} \end{pmatrix}$$



# End-effector Configuration Parameters
Example

- Most general robot arm:
  - $\mathbf{q} = \left(q_1 \ldots q_{n_j}\right)$
  - $m_e = 6$ $m_o = 6$
  - $\chi_e = \left(x, y, z, \alpha_x, \beta_y, \gamma_z\right)$ $\chi_o = \left(x, y, z, \alpha_x, \beta_y, \gamma_z\right)$
- SCARA robot arm
  - $\mathbf{q} = \left(\alpha, \beta, \gamma, \zeta\right)$
  - $m_e = 6$ $m_o = 4$
  - $\chi_e = \left(x, y, z, \alpha_x, \beta_y, \gamma_z\right)$ $\chi_o = \left(x, y, z, \gamma_z\right)$

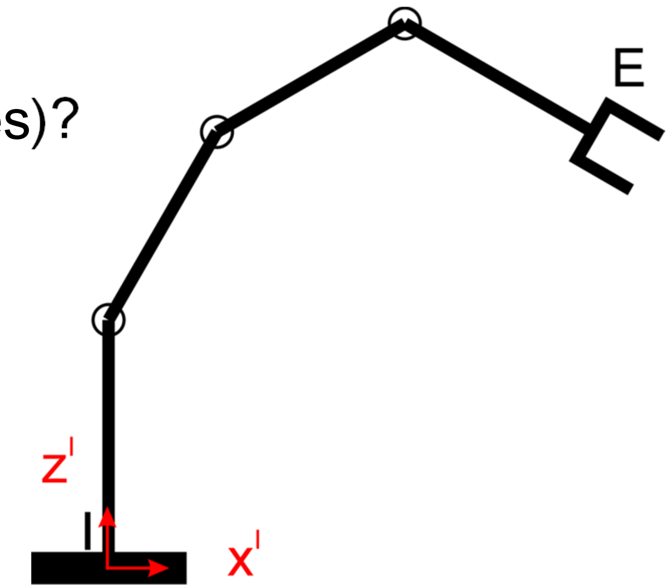



- ANYpulator: robot arm with 4 rotational joints
  - $\mathbf{q} = \left(q_1, q_2, q_3, q_4\right)$
  - $m_e = 6$ $m_o = 4$
  - $\chi_e = \left(x, y, z, \alpha_x, \beta_y, \gamma_z\right)$ $\chi_o =$

# End-effector Configuration Parameters
## Simple example

- Planar robot arm
  - 3 revolute joints
  - 1 end-effector (gripper) <= *don't consider this for the moment*

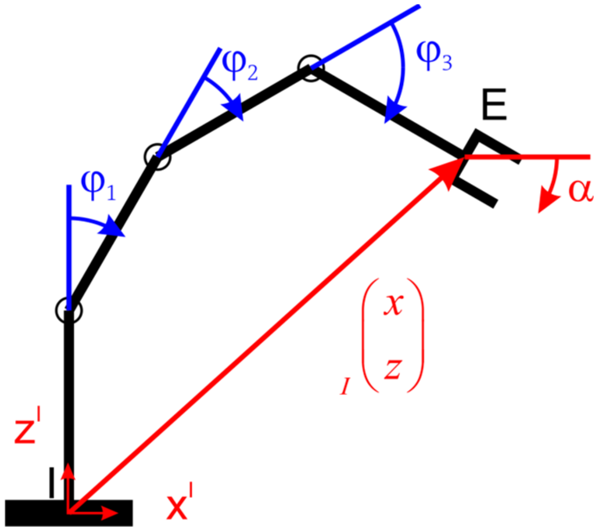
- What are the joint coordinates (generalized coordinates)?




- What are the end-effector parameters?

# Configuration Space ⇔ Joint Space

- Joint Coordinates




- Operational Coordinates

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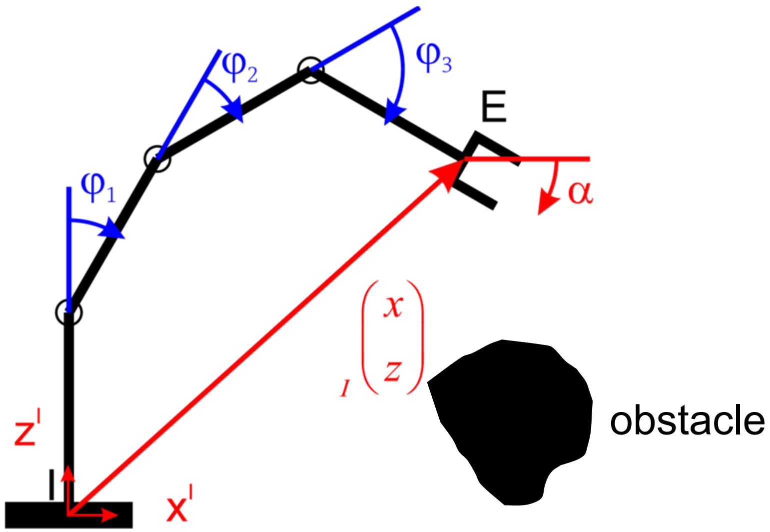
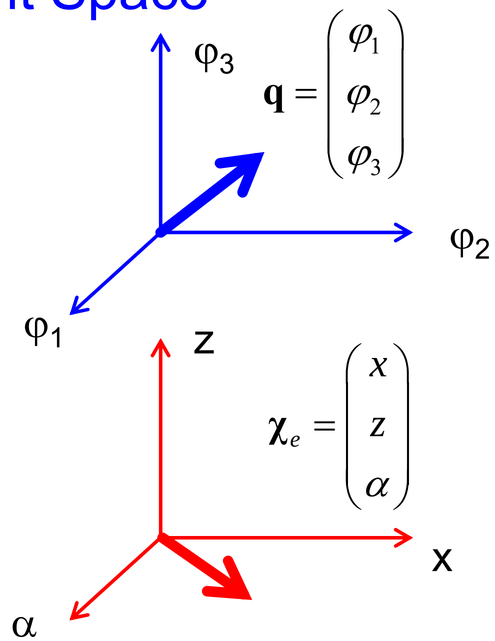
# Configuration Space ⇔ Joint Space

- Joint Coordinates => Joint Space




- Operational Coordinates => Operational Space

# Forward Kinematics

- End-effector configuration as a function of generalized coordinates

$$\chi_e = \chi_e(\mathbf{q}) \in \mathbb{R}^{n_e}$$

- For multi-body system, use transformation matrices

$$\mathbf{T}_{\mathcal{I}\mathcal{E}}(\mathbf{q}) = \mathbf{T}_{\mathcal{I}0} \cdot \left( \prod_{k=1}^{n_j} \mathbf{T}_{k-1,k}(q_k) \right) \cdot \mathbf{T}_{n_j\mathcal{E}} = \begin{bmatrix} \mathbf{C}_{\mathcal{I}\mathcal{E}}(\mathbf{q}) & {}_{\mathcal{I}}\mathbf{r}_{IE}(\mathbf{q}) \\ \mathbf{0}_{1\times 3} & 1 \end{bmatrix}$$

- Note: depending on the selected end-effector parameterization, it is not possible to analytically write down end-effector parameters!

# Forward Kinematics
## Simple example

$$\mathbf{q} = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} = \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{pmatrix}$$

- What is the end-effector configuration as a function of generalized coordinates?

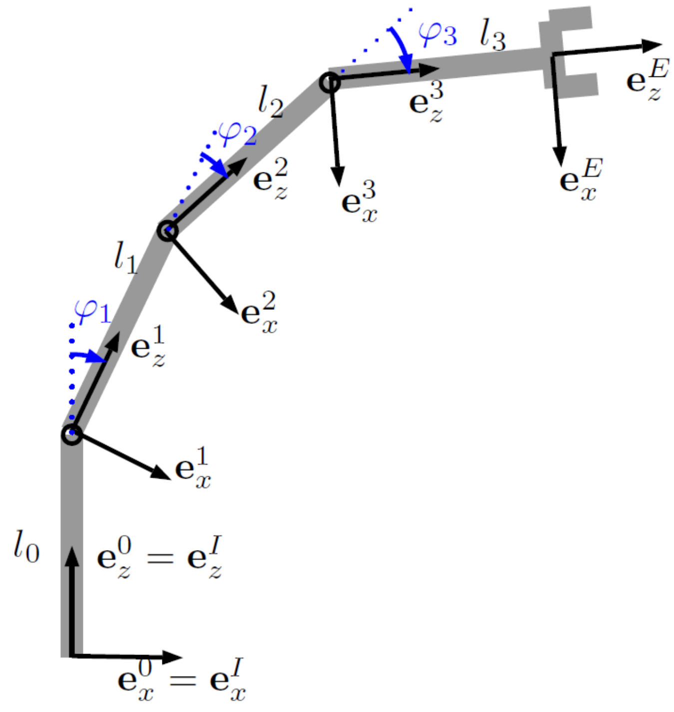
$$\mathbf{T}_{IE} = \mathbf{T}_{I0} \cdot \mathbf{T}_{01} \cdot \mathbf{T}_{12} \cdot \mathbf{T}_{23} \cdot \mathbf{T}_{3E}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 & 0 & s_1 & 0 \\ 0 & 1 & 0 & 0 \\ -s_1 & 0 & c_1 & l_0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_2 & 0 & s_2 & 0 \\ 0 & 1 & 0 & 0 \\ -s_2 & 0 & c_2 & l_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_3 & 0 & s_3 & 0 \\ 0 & 1 & 0 & 0 \\ -s_3 & 0 & c_3 & l_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & l_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \ldots = \begin{bmatrix} c_{123} & 0 & s_{123} & l_1 s_1 + l_2 s_{12} + l_3 s_{123} \\ 0 & 1 & 0 & 0 \\ -s_{123} & 0 & c_{123} & l_0 + l_1 c_1 + l_2 c_{12} + l_3 c_{123} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\chi_{eP}(\mathbf{q}) = \begin{pmatrix} x \\ z \end{pmatrix} = \boxed{\begin{pmatrix} l_1 \sin(q_1) + l_2 \sin(q_1 + q_2) + l_3 \sin(q_1 + q_2 + q_3) \\ l_0 + l_1 \cos(q_1) + l_2 \cos(q_1 + q_2) + l_3 \cos(q_1 + q_2 + q_3) \end{pmatrix}}$$

$$\chi_{eR}(\mathbf{q}) = \chi_{eR}(\mathbf{q}) = \boxed{q_1 + q_2 + q_3}$$



# Forward Differential Kinematics

## Analytical Jacobian

- Forward Kinematics $\chi_e = \begin{pmatrix} \chi_{e_P} \\ \chi_{e_R} \end{pmatrix} = \chi_e(\mathbf{q})$ $\qquad \mathbf{T}_{\mathcal{IE}}(\mathbf{q}) = \mathbf{T}_{\mathcal{I}0} \cdot \left( \prod_{k=1}^{n_j} \mathbf{T}_{k-1,k}(q_k) \right) \cdot \mathbf{T}_{n_j\mathcal{E}} = \begin{bmatrix} \mathbf{C}_{\mathcal{IE}}(\mathbf{q}) & {}_{\mathcal{I}}\mathbf{r}_{IE}(\mathbf{q}) \\ \mathbf{0}_{1\times 3} & 1 \end{bmatrix}$

- Forward **Differential** Kinematics
  - Analytic:

$$\boldsymbol{\chi}_e + \delta\boldsymbol{\chi}_e = \boldsymbol{\chi}_e(\mathbf{q} + \delta\mathbf{q}) = \boldsymbol{\chi}_e(\mathbf{q}) + \frac{\partial \boldsymbol{\chi}_e(\mathbf{q})}{\partial \mathbf{q}}\delta\mathbf{q} + O\left(\delta\mathbf{q}^2\right)$$

$$\delta\boldsymbol{\chi}_e \approx \frac{\partial \boldsymbol{\chi}_e(\mathbf{q})}{\partial \mathbf{q}}\delta\mathbf{q} = \mathbf{J}_{eA}(\mathbf{q})\,\delta\mathbf{q} \quad \text{with } \mathbf{J}_{eA} = \frac{\partial \boldsymbol{\chi}_e}{\partial \mathbf{q}} = \begin{bmatrix} \frac{\partial \chi_1}{\partial q_1} & \cdots & \frac{\partial \chi_1}{\partial q_{n_j}} \\ \vdots & \ddots & \vdots \\ \frac{\partial \chi_m}{\partial q_1} & \cdots & \frac{\partial \chi_m}{\partial q_{n_j}} \end{bmatrix}$$

$$\boxed{\dot{\boldsymbol{\chi}}_e = \mathbf{J}_{eA}(\mathbf{q})\,\dot{\mathbf{q}} \quad \text{with } \mathbf{J}_{eA}(\mathbf{q}) \in \mathbb{R}^{m_e \times n_j}}$$

# Analytical Jacobian
## Planar robot arm

- Given (from last example)

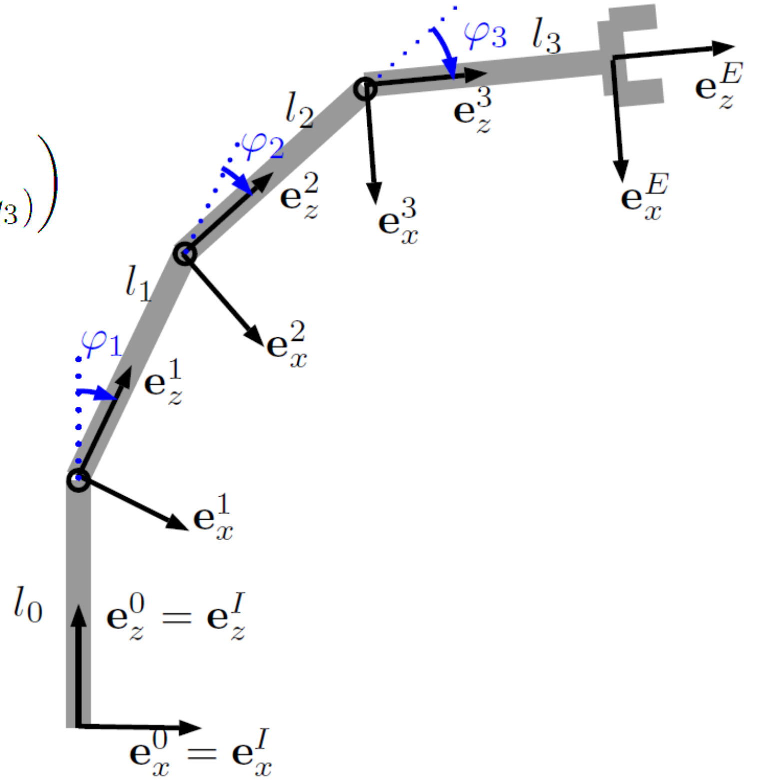
$$\chi_{eP}(\mathbf{q}) = \begin{pmatrix} x \\ z \end{pmatrix} = \begin{pmatrix} l_1 \sin(q_1) + l_2 \sin(q_1+q_2) + l_3 \sin(q_1+q_2+q_3) \\ l_0 + l_1 \cos(q_1) + l_2 \cos(q_1+q_2) + l_3 \cos(q_1+q_2+q_3) \end{pmatrix}$$

$$\chi_{eR}(\mathbf{q}) = \chi_{eR}(\mathbf{q}) = q_1 + q_2 + q_3$$

- Determine the analytical Jacobian

$$\mathbf{J}_{eAP}(\mathbf{q}) = \frac{\partial \chi_{eP}}{\partial \mathbf{q}} = \begin{bmatrix} l_1 c_1 + l_2 c_{12} + l_3 c_{123} & l_2 c_{12} + l_3 c_{213} & l_3 c_{213} \\ -l_1 s_1 - l_2 s_{12} - l_3 s_{123} & -l_2 s_{12} - l_3 s_{213} & -l_3 s_{213} \end{bmatrix} \in \mathbb{R}^{2\times 3}$$

$$\mathbf{J}_{eAR}(\mathbf{q}) = \frac{\partial \chi_{eR}}{\partial \mathbf{q}} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \in \mathbb{R}^{1\times 3}$$



# Forward Differential Kinematics

- Analytic: $\delta\boldsymbol{\chi}_e \approx \frac{\partial \boldsymbol{\chi}_e(\mathbf{q})}{\partial \mathbf{q}}\delta\mathbf{q} = \mathbf{J}_{eA}(\mathbf{q})\,\delta\mathbf{q}$ with $\mathbf{J}_{eA} = \frac{\partial \boldsymbol{\chi}_e}{\partial \mathbf{q}} = \begin{bmatrix} \frac{\partial \chi_1}{\partial q_1} & \cdots & \frac{\partial \chi_1}{\partial q_{n_j}} \\ \vdots & \ddots & \vdots \\ \frac{\partial \chi_m}{\partial q_1} & \cdots & \frac{\partial \chi_m}{\partial q_{n_j}} \end{bmatrix}$

$$\boxed{\dot{\boldsymbol{\chi}}_e = \mathbf{J}_{eA}(\mathbf{q})\,\dot{\mathbf{q}} \quad \text{with } \mathbf{J}_{eA}(\mathbf{q}) \in \mathbb{R}^{m_e \times n_j}}$$

Depending on parameterization!!

- Geometric: $\boxed{\mathbf{w}_e = \begin{pmatrix} \mathbf{v}_e \\ \boldsymbol{\omega}_e \end{pmatrix} = \mathbf{J}_{e0}(\mathbf{q})\,\dot{\mathbf{q}} \quad \text{with } \mathbf{J}_{e0}(\mathbf{q}) \in \mathbb{R}^{6 \times n_j}}$

Independent of parameterization

$$\mathbf{w}_e = \mathbf{E}_e(\boldsymbol{\chi}_e)\,\dot{\boldsymbol{\chi}}_e$$

$$\downarrow$$

$$\mathbf{J}_{e0}(\mathbf{q}) = \mathbf{E}_e(\boldsymbol{\chi})\,\mathbf{J}_{eA}(\mathbf{q})$$

- Algebra: $\mathbf{w}_C = \begin{pmatrix} \mathbf{v}_C \\ \boldsymbol{\omega}_C \end{pmatrix} = \mathbf{w}_B + \mathbf{w}_{BC}$

$$\mathbf{J}_C\dot{\mathbf{q}} = \mathbf{J}_B\dot{\mathbf{q}} + \mathbf{J}_{BC}\dot{\mathbf{q}}$$

$${}_{\mathcal{A}}\mathbf{J}_C = {}_{\mathcal{A}}\mathbf{J}_B + {}_{\mathcal{A}}\mathbf{J}_{BC}$$

# Velocity in Moving Bodies

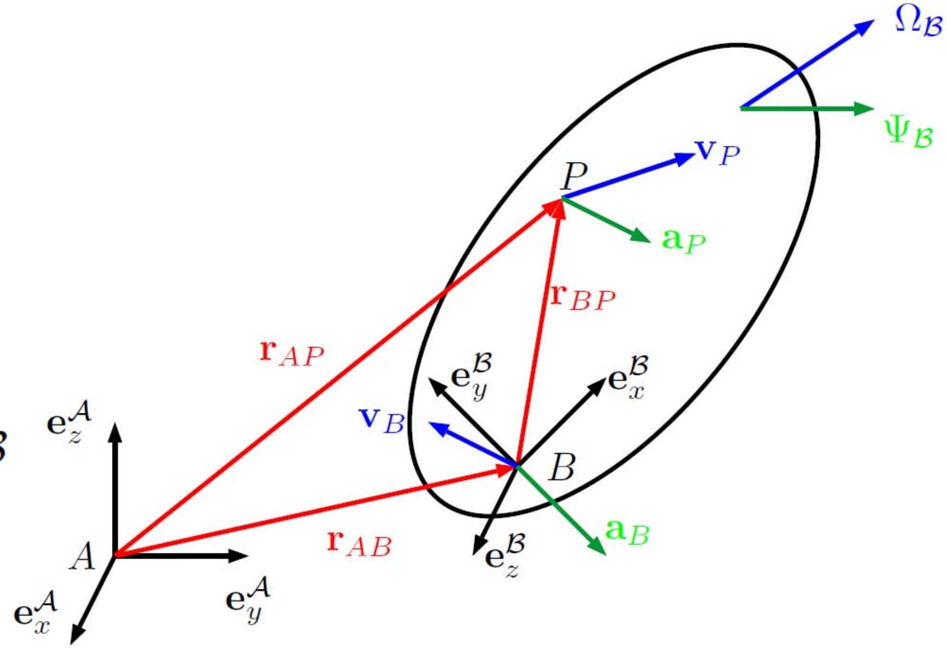
- Definitions

  $\mathbf{v}_P$: the absolute velocity of $P$

  $\mathbf{a}_P = \dot{\mathbf{v}}_P$: the absolute acceleration of $P$

  $\boldsymbol{\Omega}_\mathcal{B} = \boldsymbol{\omega}_{\mathcal{AB}}$: (absolute) angular velocity of body $\mathcal{B}$

  $\boldsymbol{\Psi}_\mathcal{B} = \dot{\boldsymbol{\Omega}}_\mathcal{B}$: (absolute) angular acceleration of body $\mathcal{B}$




- Remember the difference:
  - Velocity $\quad {}_\mathcal{C}\left(\dot{\mathbf{r}}_{AP}\right) = {}_\mathcal{C}\left(\frac{d}{dt}\mathbf{r}_{AP}\right) = {}_\mathcal{C}\,\mathbf{v}_{AP}$
  - Time derivative of coordinates: $\left({}_\mathcal{C}\dot{\mathbf{r}}_{AP}\right) = \left({}_\mathcal{C}\mathbf{r}_{AP}\right)^{\cdot} = \frac{d}{dt}\left({}_\mathcal{C}\mathbf{r}_{AP}\right)$

# Vector Differentiation in Moving Frame
## Euler differentiation rule

- For non-moving reference frames: $_{\mathcal{A}}\mathbf{v}_P = {}_{\mathcal{A}}\dot{\mathbf{r}}_{AP}$
- For moving reference frames: $\mathbf{v}_P \neq \dot{\mathbf{r}}_{AP}$
- Vector differentiation in moving frames ($\mathcal{A}$ = inertial/reference frame):

$$\begin{aligned}
{}_{\mathcal{B}}\mathbf{v}_P &= \mathbf{C}_{\mathcal{B}\mathcal{A}} \cdot \frac{d}{dt}\left(\mathbf{C}_{\mathcal{A}\mathcal{B}} \cdot {}_{\mathcal{B}}\mathbf{r}_{AP}\right) \\
&= \mathbf{C}_{\mathcal{B}\mathcal{A}} \cdot \left(\mathbf{C}_{\mathcal{A}\mathcal{B}} \cdot {}_{\mathcal{B}}\dot{\mathbf{r}}_{AP} + \dot{\mathbf{C}}_{\mathcal{A}\mathcal{B}} \cdot {}_{\mathcal{B}}\mathbf{r}_{AP}\right) \\
&= \mathbf{C}_{\mathcal{B}\mathcal{A}} \cdot \left(\mathbf{C}_{\mathcal{A}\mathcal{B}} \cdot {}_{\mathcal{B}}\dot{\mathbf{r}}_{AP} + [{}_{\mathcal{A}}\boldsymbol{\omega}_{\mathcal{A}\mathcal{B}}]_\times \cdot \mathbf{C}_{\mathcal{A}\mathcal{B}} \cdot {}_{\mathcal{B}}\mathbf{r}_{AP}\right) \\
&= {}_{\mathcal{B}}\dot{\mathbf{r}}_{AP} + \mathbf{C}_{\mathcal{B}\mathcal{A}} \cdot [{}_{\mathcal{A}}\boldsymbol{\omega}_{\mathcal{A}\mathcal{B}}]_\times \cdot \mathbf{C}_{\mathcal{A}\mathcal{B}} \cdot {}_{\mathcal{B}}\mathbf{r}_{AP} \\
&= {}_{\mathcal{B}}\dot{\mathbf{r}}_{AP} + {}_{\mathcal{B}}\boldsymbol{\omega}_{\mathcal{A}\mathcal{B}} \times {}_{\mathcal{B}}\mathbf{r}_{AP}
\end{aligned}$$

$$[{}_{\mathcal{A}}\omega_{\mathcal{A}\mathcal{B}}]_\times = \dot{\mathbf{C}}_{\mathcal{A}\mathcal{B}}\mathbf{C}_{\mathcal{A}\mathcal{B}}^T$$

$$[{}_{\mathcal{B}}\boldsymbol{\omega}_{\mathcal{A}\mathcal{B}}]_\times = \mathbf{C}_{\mathcal{B}\mathcal{A}} \cdot [{}_{\mathcal{A}}\boldsymbol{\omega}_{\mathcal{A}\mathcal{B}}]_\times \cdot \mathbf{C}_{\mathcal{A}\mathcal{B}}$$

# Velocity in Moving Bodies
## Rigid body formulation

- Apply transformation rule as learned before

$$ {}_{\mathcal{A}}\mathbf{r}_{AP} = {}_{\mathcal{A}}\mathbf{r}_{AB} + {}_{\mathcal{A}}\mathbf{r}_{BP} = {}_{\mathcal{A}}\mathbf{r}_{AB} + \mathbf{C}_{\mathcal{AB}} \cdot {}_{\mathcal{B}}\mathbf{r}_{BP} $$

- Differentiate with respect to time

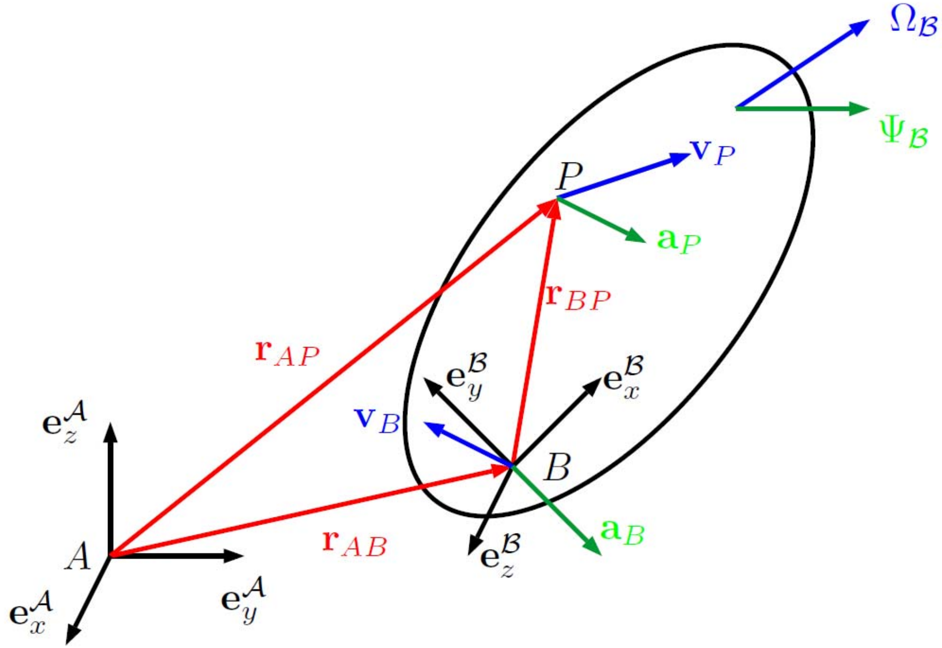
$$ {}_{\mathcal{A}}\dot{\mathbf{r}}_{AP} = {}_{\mathcal{A}}\dot{\mathbf{r}}_{AB} + \mathbf{C}_{\mathcal{AB}} \cdot {}_{\mathcal{B}}\dot{\mathbf{r}}_{BP} + \dot{\mathbf{C}}_{\mathcal{AB}} \cdot {}_{\mathcal{B}}\mathbf{r}_{BP} $$

- Substitute $\dot{\mathbf{C}}_{\mathcal{AB}} = [{}_{\mathcal{A}}\boldsymbol{\omega}_{\mathcal{AB}}]_{\times} \cdot \mathbf{C}_{\mathcal{AB}}$

- Rigid body formulation

$$ \boxed{\begin{aligned} {}_{\mathcal{A}}\dot{\mathbf{r}}_{AP} &= {}_{\mathcal{A}}\dot{\mathbf{r}}_{AB} + [{}_{\mathcal{A}}\boldsymbol{\omega}_{\mathcal{AB}}]_{\times} \cdot \mathbf{C}_{\mathcal{AB}} \cdot {}_{\mathcal{B}}\mathbf{r}_{BP} \\ &= {}_{\mathcal{A}}\dot{\mathbf{r}}_{AB} + {}_{\mathcal{A}}\boldsymbol{\omega}_{\mathcal{AB}} \times {}_{\mathcal{A}}\mathbf{r}_{BP} \end{aligned}} $$

$$ \mathbf{v}_P = \mathbf{v}_B + \boldsymbol{\Omega} \times \mathbf{r}_{BP} $$



# Geometric Jacobian Derivation

- Rigid body formulation at a single element

$$\dot{\mathbf{r}}_{Ik} = \dot{\mathbf{r}}_{I(k-1)} + \boldsymbol{\omega}_{\mathcal{I}(k-1)} \times \mathbf{r}_{(k-1)k}$$

- Apply this to all bodies

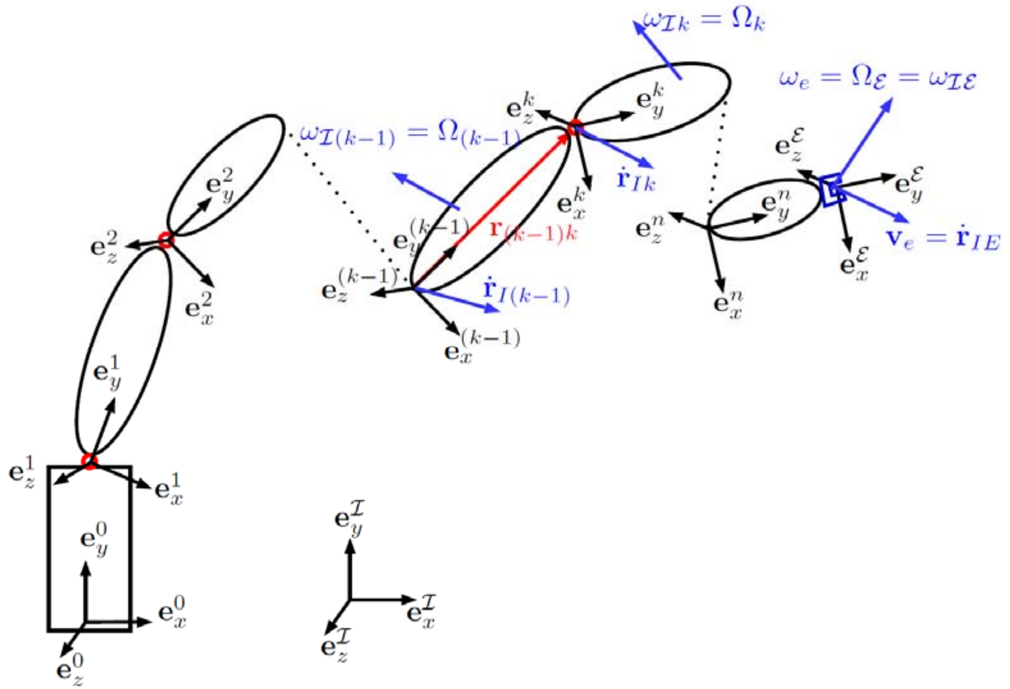
$$\dot{\mathbf{r}}_{IE} = \sum_{k=1}^{n} \boldsymbol{\omega}_{\mathcal{I},k} \times \mathbf{r}_{k(k+1)}$$

- Angular velocity propagation

$$\left.\begin{aligned} \boldsymbol{\omega}_{\mathcal{I}(k)} &= \boldsymbol{\omega}_{\mathcal{I}(k-1)} + \boldsymbol{\omega}_{(k-1)k} \\ \text{with} \quad \boldsymbol{\omega}_{(k-1)k} &= \mathbf{n}_k \dot{q}_k \end{aligned}\right\} \quad \boldsymbol{\omega}_{\mathcal{I}k} = \sum_{i=1}^{k} \mathbf{n}_i \dot{q}_i$$

- …get the end-effector velocity

$$\dot{\mathbf{r}}_{IE} = \sum_{k=1}^{n} \left\{ \sum_{i=1}^{k} (\mathbf{n}_i \dot{q}_i) \times \mathbf{r}_{k(k+1)} \right\} = \sum_{k=1}^{n} \mathbf{n}_k \dot{q}_k \times \mathbf{r}_{k(n+1)}$$



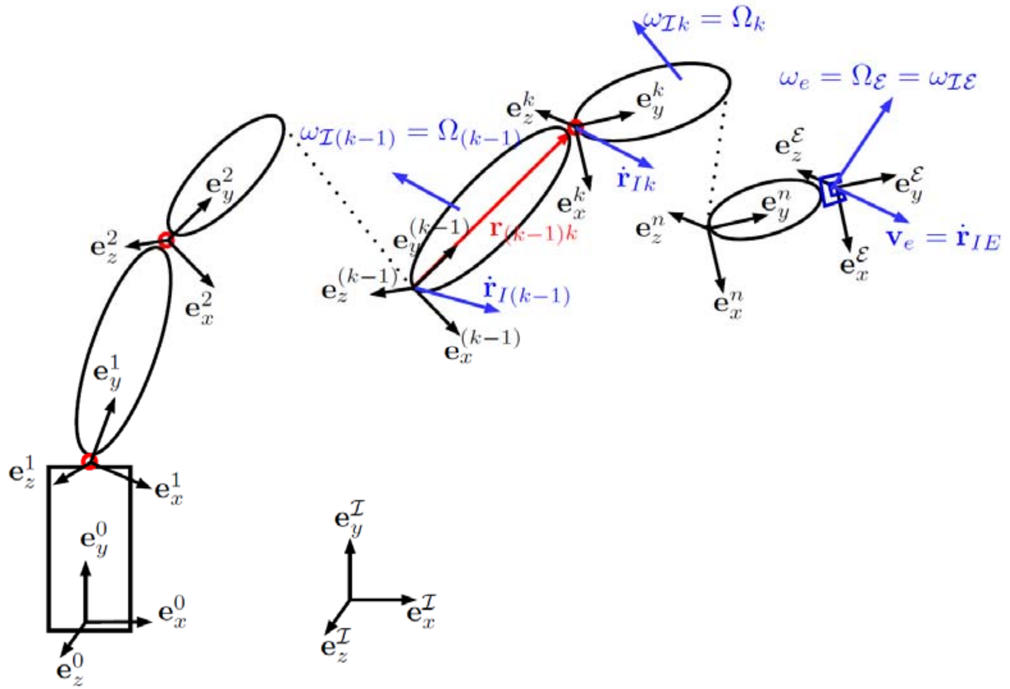
# Geometric Jacobian Derivation

- Position Jacobian $\dot{\mathbf{r}}_{IE} = \sum_{k=1}^{n} \mathbf{n}_k \dot{q}_k \times \mathbf{r}_{k(n+1)}$

$$\dot{\mathbf{r}}_{IE} = \underbrace{\begin{bmatrix} \mathbf{n}_1 \times \mathbf{r}_{1(n+1)} & \mathbf{n}_2 \times \mathbf{r}_{2(n+1)} & \dots & \mathbf{n}_n \times \mathbf{r}_{n(n+1)} \end{bmatrix}}_{\mathbf{J}_{e0_P}} \begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_n \end{pmatrix}$$

- Rotation Jacobian from $\boldsymbol{\omega}_{\mathcal{I}k} = \sum_{i=1}^{k} \mathbf{n}_i \dot{q}_i$

$$\boldsymbol{\omega}_{\mathcal{I}\mathcal{E}} = \sum_{i=1}^{n} \mathbf{n}_i \dot{q}_i = \underbrace{\begin{bmatrix} \mathbf{n}_1 & \mathbf{n}_2 & \dots & \mathbf{n}_n \end{bmatrix}}_{\mathbf{J}_{e0_R}} \begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_n \end{pmatrix}$$




$$\Rightarrow \quad {}_{\mathcal{I}}\mathbf{J}_{e0} = \begin{bmatrix} {}_{\mathcal{I}}\mathbf{J}_{e0_P} \\ {}_{\mathcal{I}}\mathbf{J}_{e0_R} \end{bmatrix} = \begin{bmatrix} {}_{\mathcal{I}}\mathbf{n}_1 \times {}_{\mathcal{I}}\mathbf{r}_{1(n+1)} & {}_{\mathcal{I}}\mathbf{n}_2 \times {}_{\mathcal{I}}\mathbf{r}_{2(n+1)} & \dots & {}_{\mathcal{I}}\mathbf{n}_n \times {}_{\mathcal{I}}\mathbf{r}_{n(n+1)} \\ {}_{\mathcal{I}}\mathbf{n}_1 & {}_{\mathcal{I}}\mathbf{n}_2 & \dots & {}_{\mathcal{I}}\mathbf{n}_n \end{bmatrix}$$

$$\boxed{{}_{\mathcal{I}}\mathbf{n}_k = \mathbf{C}_{\mathcal{I}(k-1)} \cdot {}_{(k-1)}\mathbf{n}_k}$$

# Geometric Jacobian
## Planar robot arm

- Preparation: determine the rotation matrices

$$\mathbf{C}_{\mathcal{I}1} = \begin{bmatrix} c_1 & 0 & s_1 \\ 0 & 1 & 0 \\ -s_1 & 0 & c_1 \end{bmatrix} \qquad \mathbf{C}_{\mathcal{I}2} = \mathbf{C}_{\mathcal{I}1} \cdot \begin{bmatrix} c_2 & 0 & s_2 \\ 0 & 1 & 0 \\ -s_2 & 0 & c_2 \end{bmatrix} = \begin{bmatrix} c_{12} & 0 & s_{12} \\ 0 & 1 & 0 \\ -s_{12} & 0 & c_{12} \end{bmatrix} \qquad \mathbf{C}_{\mathcal{I}3} = \ \ldots$$

- Determine the rotation axes
  - Locally $\quad {}_0\mathbf{n}_1 = {}_1\mathbf{n}_2 = {}_2\mathbf{n}_3 = \mathbf{e}_y$

Inertial frame

$${}_\mathcal{I}\mathbf{n}_1 = {}_0\mathbf{n}_1 = \mathbf{e}_y$$
$${}_\mathcal{I}\mathbf{n}_2 = \mathbf{C}_{I1} \cdot {}_1\mathbf{n}_2 = \mathbf{e}_y$$
$${}_\mathcal{I}\mathbf{n}_3 = \mathbf{C}_{I2} \cdot {}_2\mathbf{n}_3 = \mathbf{e}_y$$

- Determine the position vectors

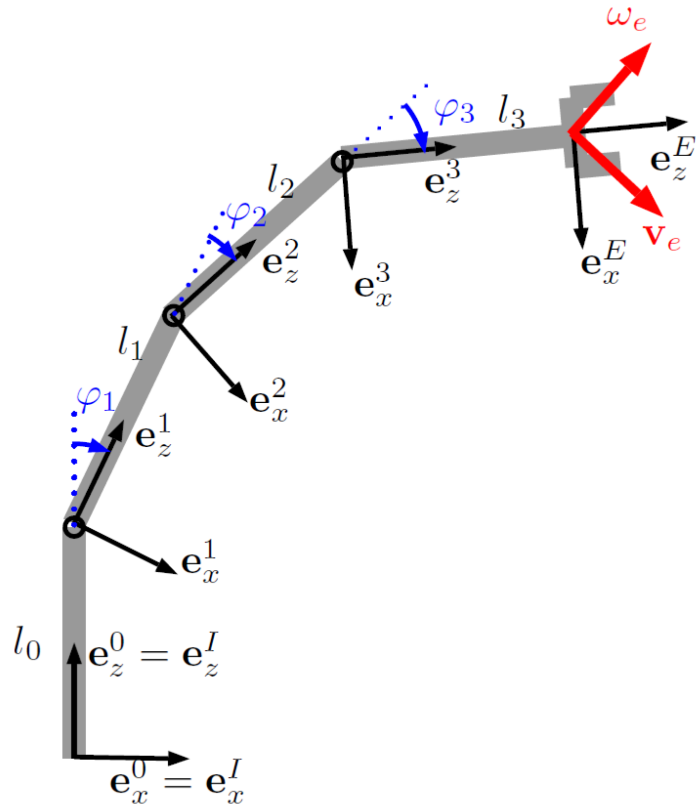
$${}_\mathcal{I}\mathbf{r}_{1E} = {}_\mathcal{I}\mathbf{r}_{12} + {}_\mathcal{I}\mathbf{r}_{23} + {}_\mathcal{I}\mathbf{r}_{3E} = \mathbf{C}_{I1} \cdot {}_1\mathbf{r}_{12} + \mathbf{C}_{I2} \cdot {}_2\mathbf{r}_{23} + \mathbf{C}_{I3} \cdot {}_3\mathbf{r}_{3E} = l_1 \begin{pmatrix} s_{q_1} \\ 0 \\ c_{q_1} \end{pmatrix} + l_2 \begin{pmatrix} s_{12} \\ 0 \\ c_{12} \end{pmatrix} + l_3 \begin{pmatrix} s_{123} \\ 0 \\ c_{123} \end{pmatrix}$$

$${}_\mathcal{I}\mathbf{r}_{2E} = {}_\mathcal{I}\mathbf{r}_{23} + {}_\mathcal{I}\mathbf{r}_{3E} = \ldots \qquad {}_\mathcal{I}\mathbf{r}_{3E} = \ \ldots$$

- Get the Jacobian

$${}_\mathcal{I}\mathbf{J}_{e0_P} = \begin{bmatrix} {}_\mathcal{I}\mathbf{n}_1 \times {}_\mathcal{I}\mathbf{r}_{1E} & {}_\mathcal{I}\mathbf{n}_2 \times {}_\mathcal{I}\mathbf{r}_{2E} & {}_\mathcal{I}\mathbf{n}_3 \times {}_\mathcal{I}\mathbf{r}_{3E} \end{bmatrix} = \begin{bmatrix} l_1c_1 + l_2c_{12} + l_3c_{123} & l_2c_{12} + l_3c_{123} & l_3c_{123} \\ 0 & 0 & 0 \\ -l_1s_1 - l_2s_{12} - l_3c_{123} & -l_2s_{12} - l_3s_{123} & -l_3s_{123} \end{bmatrix}$$

$${}_\mathcal{I}\mathbf{J}_{e0_R} = \begin{bmatrix} {}_\mathcal{I}\mathbf{n}_1 & {}_\mathcal{I}\mathbf{n}_2 & {}_\mathcal{I}\mathbf{n}_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$



# Recapitulation
## Analytical and Kinematic Jacobian

- Analytical Jacobian

$$\dot{\chi}_e = \mathbf{J}_{eA}(\mathbf{q})\,\dot{\mathbf{q}}$$

$$\mathbf{J}_{e0}(\mathbf{q}) = \mathbf{E}_e(\chi)\,\mathbf{J}_{eA}(\mathbf{q})$$

  - Relates time-derivatives of config. parameters to generalized velocities
  - Depending on selected parameterization (mainly rotation) in 3D $\Delta\chi \Leftrightarrow \Delta\mathbf{q}$
    *Note: there exist no "rotation angle"*
  - Mainly used for numeric algorithms

- Geometric (or basic) Jacobian

$$\mathbf{w}_e = \begin{pmatrix} \mathbf{v}_e \\ \boldsymbol{\omega}_e \end{pmatrix} = \mathbf{J}_{e0}(\mathbf{q})\,\dot{\mathbf{q}}$$

  - Relates end-effector velocity to generalized velocities
  - Unique for every robot
  - Used in most cases

# Importance of Jacobian

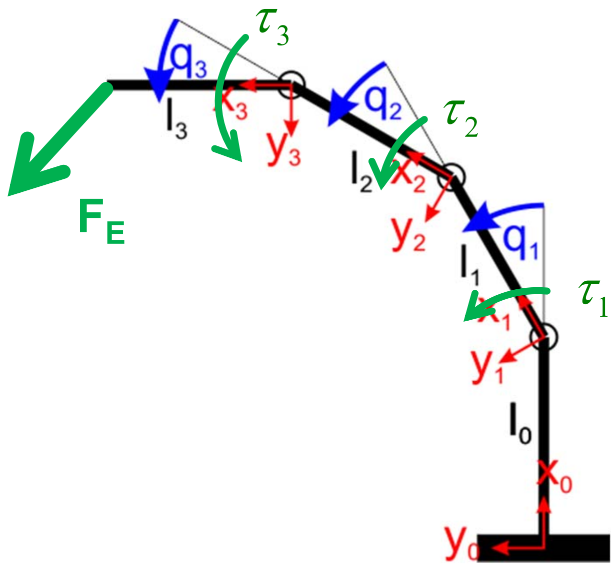
- Kinematics (mapping of changes from joint to task space)
  - Inverse kinematics control
  - Resolve redundancy problems
  - Express contact constraints
- Statics (and later also dynamics)
  - Principle of virtual work
    - Variations in work must cancel for all virtual displacement
    - Internal forces of ideal joint don't contribute

$$\delta W = \sum_i \mathbf{f}_i \mathbf{x}_i = \boldsymbol{\tau}^T \delta \mathbf{q} + \left(-\mathbf{F}_E\right)^T \delta \mathbf{x}_E$$

$$= \boldsymbol{\tau}^T \delta \mathbf{q} + \left(-\mathbf{F}_E\right)^T \mathbf{J} \delta \mathbf{q} = 0 \quad \forall \delta \mathbf{q}$$

➢ Dual problem from principle of virtual work

$$\boxed{\begin{aligned} \dot{\mathbf{x}} &= \mathbf{J}\dot{\mathbf{q}} \\ \boldsymbol{\tau} &= \mathbf{J}^T \mathbf{F} \end{aligned}}$$



# Floating Base Kinematics

**151-0851-00 V**

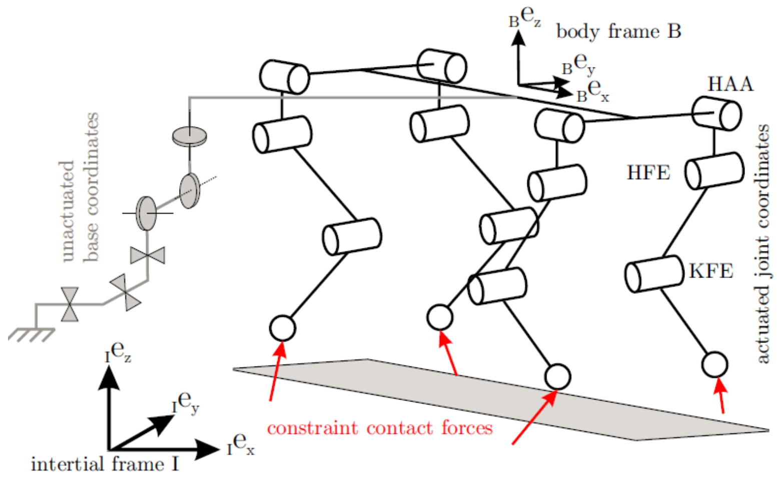
lecture: CAB G11 Tuesday 10:15 – 12:00, every week

exercise: HG E1.2 Wednesday 8:15 – 10:00, according to schedule (about every 2nd week)

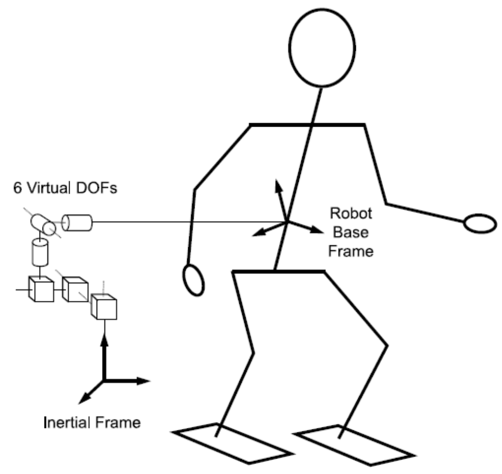
office hour: LEE H303 Friday 12.15 – 13.00

Marco Hutter, Roland Siegwart, and Thomas Stastny

# Floating Base Systems
## Kinematics




(a) Quadruped




(b) Humanoid

- Generalized coordinates

$$\mathbf{q} = \begin{pmatrix} \mathbf{q}_b \\ \mathbf{q}_j \end{pmatrix} \quad \text{with} \quad \mathbf{q}_b = \begin{pmatrix} \mathbf{q}_{b_P} \\ \mathbf{q}_{b_R} \end{pmatrix} \in \mathbb{R}^3 \times SO(3)$$

- Generalized velocities and accelerations?
  - Time derivatives $\dot{\mathbf{q}}, \ddot{\mathbf{q}}$ depend on parameterization
  - Often $\mathbf{u} = \begin{pmatrix} {}_I\mathbf{v}_B \\ \boxed{B}\boldsymbol{\omega}_{IB} \\ \dot{\varphi}_1 \\ \vdots \\ \dot{\varphi}_{n_j} \end{pmatrix} \in \mathbb{R}^{6+n_j} = \mathbb{R}^{n_u} \qquad \dot{\mathbf{u}} = \begin{pmatrix} {}_I\mathbf{a}_B \\ \boxed{B}\boldsymbol{\psi}_{IB} \\ \ddot{\varphi}_1 \\ \vdots \\ \ddot{\varphi}_{n_j} \end{pmatrix} \in \mathbb{R}^{6+n_j}$
  - Linear mapping $\mathbf{u} = \mathbf{E}_{fb} \cdot \dot{\mathbf{q}}$, with $\mathbf{E}_{fb} = \begin{bmatrix} \mathbb{I}_{3\times3} & 0 & 0 \\ 0 & \mathbf{E}_{\boldsymbol{\chi}_R} & 0 \\ 0 & 0 & \mathbb{I}_{n_j \times n_j} \end{bmatrix}$

# Floating Base Systems
## Differential kinematics

- Position of an arbitrary point on the robot

$$
{}_{\mathcal{I}}\mathbf{r}_{IQ}(\mathbf{q}) = \boxed{{}_{\mathcal{I}}\mathbf{r}_{IB}(\mathbf{q})} + \boxed{\mathbf{C}_{\mathcal{IB}}(\mathbf{q})} \cdot \boxed{{}_{\mathcal{B}}\mathbf{r}_{BQ}(\mathbf{q})}
$$

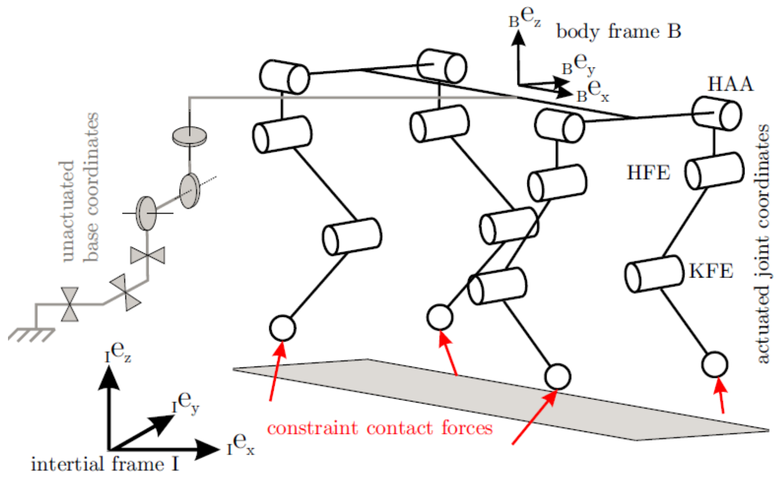
$$
{}_{\mathcal{I}}\mathbf{r}_{IB}(\mathbf{q}_b) \quad \mathbf{C}_{\mathcal{IB}}(\mathbf{q}_b) \quad {}_{\mathcal{B}}\mathbf{r}_{BQ}(\mathbf{q}_j)
$$

- Velocity of this point

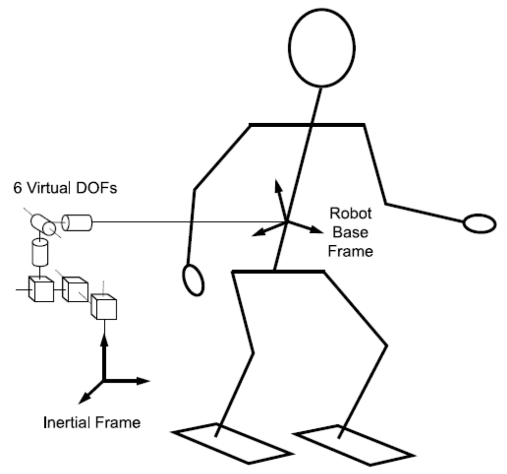
$$
\begin{aligned}
{}_{\mathcal{I}}\mathbf{v}_Q &= {}_{\mathcal{I}}\mathbf{v}_B + \dot{\mathbf{C}}_{\mathcal{IB}} \cdot {}_{\mathcal{B}}\mathbf{r}_{BQ} + \mathbf{C}_{\mathcal{IB}} \cdot {}_{\mathcal{B}}\dot{\mathbf{r}}_{BQ} \\
&= {}_{\mathcal{I}}\mathbf{v}_B + \mathbf{C}_{\mathcal{IB}} \cdot [{}_{\mathcal{B}}\boldsymbol{\omega}_{\mathcal{IB}}]_\times \cdot {}_{\mathcal{B}}\mathbf{r}_{BQ} + \mathbf{C}_{\mathcal{IB}} \cdot {}_{\mathcal{B}}\dot{\mathbf{r}}_{BQ} \\
&= {}_{\mathcal{I}}\mathbf{v}_B - \mathbf{C}_{\mathcal{IB}} \cdot [{}_{\mathcal{B}}\mathbf{r}_{BQ}]_\times \cdot {}_{\mathcal{B}}\boldsymbol{\omega}_{\mathcal{IB}} + \mathbf{C}_{\mathcal{IB}} \cdot {}_{\mathcal{B}}\dot{\mathbf{r}}_{BQ} \\
&= {}_{\mathcal{I}}\mathbf{v}_B - \mathbf{C}_{\mathcal{IB}} \cdot [{}_{\mathcal{B}}\mathbf{r}_{BQ}]_\times \cdot {}_{\mathcal{B}}\boldsymbol{\omega}_{\mathcal{IB}} + \mathbf{C}_{\mathcal{IB}} \cdot {}_{\mathcal{B}}\mathbf{J}_{P_{q_j}}(\mathbf{q}_j) \cdot \dot{\mathbf{q}}_j \\
&= \underbrace{\boxed{\begin{bmatrix} \mathbb{I}_{3\times3} & -\mathbf{C}_{\mathcal{IB}} \cdot [{}_{\mathcal{B}}\mathbf{r}_{BQ}]_\times & \mathbf{C}_{\mathcal{IB}} \cdot {}_{\mathcal{B}}\mathbf{J}_{P_{q_j}}(\mathbf{q}_j) \end{bmatrix}}}_{{}_{\mathcal{I}}\mathbf{J}_Q(\mathbf{q})} \cdot \mathbf{u}
\end{aligned}
$$

with $\mathbf{u} = \begin{pmatrix} {}_{\mathcal{I}}\mathbf{v}_B \\ {}_{\mathcal{B}}\boldsymbol{\omega}_{IB} \\ \dot{\varphi}_1 \\ \vdots \\ \dot{\varphi}_{n_j} \end{pmatrix}$

$$
[{}_{\mathcal{B}}\boldsymbol{\omega}_{\mathcal{IB}}]_\times = \mathbf{C}_{\mathcal{BI}} [{}_{\mathcal{I}}\boldsymbol{\omega}_{\mathcal{IB}}]_\times \mathbf{C}_{\mathcal{BI}}^T
= \mathbf{C}_{\mathcal{IB}}^T \dot{\mathbf{C}}_{\mathcal{IB}} \mathbf{C}_{\mathcal{IB}}^T \mathbf{C}_{\mathcal{IB}} = \mathbf{C}_{\mathcal{IB}}^T \dot{\mathbf{C}}_{\mathcal{IB}}
$$




(a) Quadruped




(b) Humanoid

# Contact Constraints

- A contact point $C_i$ is not allowed to move:

$$_{\mathcal{I}}\mathbf{r}_{IC_i} = const, \quad _{\mathcal{I}}\dot{\mathbf{r}}_{IC_i} = {}_{\mathcal{I}}\ddot{\mathbf{r}}_{IC_i} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

- Constraint as a function of generalized coordinates:

$$_{\mathcal{I}}\mathbf{J}_{C_i}\mathbf{u} = \mathbf{0}, \qquad _{\mathcal{I}}\mathbf{J}_{C_i}\dot{\mathbf{u}} + {}_{\mathcal{I}}\dot{\mathbf{J}}_{C_i}\mathbf{u} = \mathbf{0}$$

- Stack of constraints

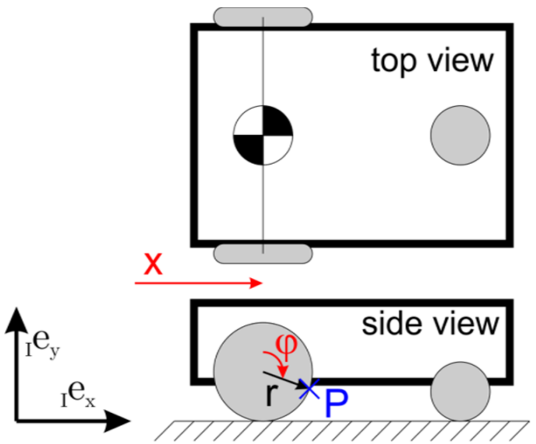
$$\mathbf{J}_c = \begin{bmatrix} \mathbf{J}_{C_1} \\ \vdots \\ \mathbf{J}_{C_{n_c}} \end{bmatrix} \in \mathbb{R}^{3n_c \times n_n}$$

# Contact Constraint
## Wheeled vehicle simple example

- Contact constraints
  - Point on wheel $${}_{\mathcal{I}}\mathbf{r}_{IP} = \begin{pmatrix} x + r\sin(\varphi) \\ r + r\cos(\varphi) \\ 0 \end{pmatrix}$$
  - Jacobian $${}_{\mathcal{I}}\mathbf{J}_{P} = \begin{bmatrix} 1 & r\cos(\varphi) \\ 0 & -r\sin(\varphi) \\ 0 & 0 \end{bmatrix}$$
  - Contact constraints

$${}_{\mathcal{I}}\dot{\mathbf{r}}_{IP}\big|_{\varphi=\pi} = {}_{\mathcal{I}}\mathbf{J}_{P}\big|_{\varphi=\pi}\dot{\mathbf{q}} = \begin{bmatrix} 1 & -r \\ 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{pmatrix} \dot{x} \\ \dot{\varphi} \end{pmatrix} = \mathbf{0}$$

=> Rolling condition $\dot{x} - r\dot{\varphi} = 0$




$$\mathbf{q} = \begin{pmatrix} x \\ \varphi \end{pmatrix}$$ Un-actuated base / Actuated joints

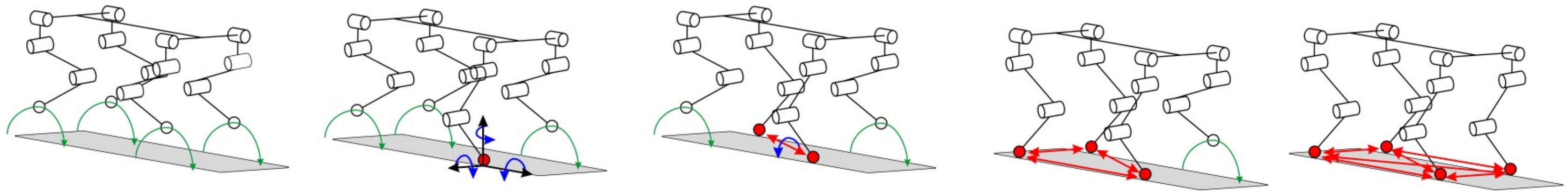
# Properties of Contact Jacobian

- Contact Jacobian tells us, how a system can move.
  - Separate stacked Jacobian $\mathbf{J}_c = \begin{bmatrix} \boxed{\mathbf{J}_{c,b}} & \mathbf{J}_{c,j} \end{bmatrix} = \begin{bmatrix} \frac{\partial \mathbf{r}_c}{\partial \mathbf{q}_b} & \frac{\partial \mathbf{r}_c}{\partial \mathbf{q}_j} \end{bmatrix} \in \mathbb{R}^{n_c \times (n_b + n_j)}$

    relation between base motion and constraints
  - Base is fully controllable if $rank(\mathbf{J}_{c,b}) = 6$
  - Nr of kinematic constraints for joint actuators: $rank(\mathbf{J}_c)$ - $rank(\mathbf{J}_{c,b})$
- Generalized coordinates DON'T correspond to the degrees of freedom
  - Contact constraints!
- Minimal coordinates (= correspond to degrees of freedom)
  - Require to switch the set of coordinates depending on contact state (=> never used)

# Quadrupedal Robot with Point Feet

- Floating base system with 12 actuated joint and 6 base coordinates (18DoF)



| | | | | | |
|---|---|---|---|---|---|
| Total constraints | 0 | 3 | 6 | 9 | 12 |
| Internal constraints | 0 | 0 | 1 | 3 | 6 |
| Uncontrollable DoFs | 6 | 3 | 1 | 0 | 0 |

# Outlook

- Exercise TOMORROW
  - Differential Kinematics
  - Use it as extended office hour!

- Next Lecture
  - Script Section 2.9 (Kinematic Control Methods)
  - Inverse Kinematics
  - Inverse Differential Kinematics