Problem 1

1.1) Given line $y = mx + b$

$$\Rightarrow \quad mx - y + b = 0$$

multiply the line W we get

$$mxW - yW + bW = 0 \qquad \text{— (1)}$$

let $X = xW$

$Y = yW$

then eq.(1) can be written as

$$mX - Y + bW = 0$$

This can be written as

$$\begin{bmatrix} m & -1 & b \end{bmatrix} \begin{bmatrix} X \\ Y \\ W \end{bmatrix} = 0$$

$$\bar{L}^T \bar{X} = 0$$

Where $\bar{L} = \begin{bmatrix} m \\ -1 \\ b \end{bmatrix}$ and $\bar{X} = \begin{bmatrix} X \\ Y \\ W \end{bmatrix}$

$$\bar{X} = \begin{bmatrix} X \\ Y \\ W \end{bmatrix} = \begin{bmatrix} X/W \\ Y/W \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$ } Proving that the point (x, y) passes through original line

1.2) $(3,5)$ on 2D plane can be written as $(3,5,1)$ in homogeneous representation

1) let $z=2$ then $(3,5,1)$ can be written as $(6,10,2)$
Since multiplying with constant leads to same point in homogeneous space

2) let $z=-2$ then $(3,5,1)$ is same as $(-6,-10,-2)$

Therefore $$(x,y,z) = \begin{cases} (6,10,2), & z>0 \\ (-6,-10,-2), & z<0 \end{cases}$$

1.3) let $a_1x + b_1y + c_1 = 0$ be two lines, in the normal space
$a_2x + b_2y + c_2 = 0$

Intersection point for these two lines is given by

$$\begin{matrix} x & y & 1 & \\ b_1 & c_1 & a_1 & b_1 \\ b_2 & c_2 & a_2 & b_2 \end{matrix} \Rightarrow \frac{x}{b_1c_2 - c_1b_2} = \frac{y}{c_1a_2 - a_1c_2} = \frac{1}{a_1b_2 - b_1a_2}$$

$$(x, y) = \left(\frac{b_1c_2 - c_1b_2}{a_1b_2 - b_1a_2}, \frac{c_1a_2 - a_1c_2}{a_1b_2 - b_1a_2} \right) \quad — (1)$$

The above lines can be represented in hetorogeneous space as

$$L_1 = \begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix} \quad L_2 = \begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix}$$

$l_1 \times l_2$ is a vector that is perpendicular to L_1 and L_2 and having modulus equal to 1

let $K = [x_1, y_1, z_1]^T = L_1 \times L_2$

$L_1 \cdot K = 0$ and $L_2 \cdot K = 0$ and $||K|| = 1$

$a_1x_1 + b_1y_1 + c_1z_1 = 0$, $a_2x_1 + b_2y_1 + c_2z_1 = 0$, $x_1^2 + y_1^2 + z_1^2 = 1$

Solving these we get
$$K = \begin{bmatrix} b_1c_2 - c_1b_2 \\ c_1a_2 - a_1c_2 \\ a_1b_2 - b_1a_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$$

Since $K = L_1 \times L_2$ in homogeneous space we can divide by z_1

to get
$$L_1 \times L_2 = \begin{bmatrix} (b_1c_2 - c_1b_2) / (a_1b_2 - b_1a_2) \\ (c_1a_2 - a_1c_2) / (a_1b_2 - b_1a_2) \\ 1 \end{bmatrix} \quad — (2)$$

From (1) and (2) we showed that homogeneous lines L_1 and L_2 intersection is given by $L_1 \times L_2$

1.4) $x+y-5=0 \Rightarrow L_1^T x = 0$ where $L_1 = \begin{bmatrix} 1 \\ 1 \\ -5 \end{bmatrix}$

$4x-5y+7=0 \Rightarrow L_2^T x = 0$ where $L_2 = \cancel{\begin{bmatrix} -18 \\ -27 \\ -9 \end{bmatrix}} \begin{bmatrix} 4 \\ -5 \\ 7 \end{bmatrix}$

$$L_1 \times L_2 = \begin{bmatrix} b_1c_2 - c_1b_2 \\ c_1a_2 - a_1c_2 \\ a_1b_2 - b_1a_2 \end{bmatrix}$$ where $(a_1,b_1,c_1) = (1,1,-5)$, $(a_2,b_2,c_2) = (4,-5,7)$

$\rightarrow$ derived in previous problem

$$= \begin{bmatrix} 7-(-5)(-5) \\ -20-7 \\ -5-4 \end{bmatrix} = \begin{bmatrix} -18 \\ -27 \\ -9 \end{bmatrix}$$

Dividing by -9 we get $L_1 \times L_2 = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$

Therefore $(2,3)$ is the intersection point of two given lines

1.5) Given line $ax+by+c=0$

let $ax+by+c_1=0$ be a parallel line

Intersection point is given by $L_1 \times L_2 = \begin{bmatrix} bc_1 - cb \\ ca - ac_1 \\ ab - ab \end{bmatrix} = \begin{bmatrix} bc_1 - cb \\ ca - ac_1 \\ 0 \end{bmatrix}$

If $c_1 = 0 \Rightarrow L_1 \times L_2 = \begin{bmatrix} -cb \\ ca \\ 0 \end{bmatrix} = \begin{bmatrix} b \\ -a \\ 0 \end{bmatrix}$

There are multiple possibilities of first two rows in $L_1 \times L_2$ as the third row is zero. i.e. In homogenous space parallel lines interrect ~~but~~ and has a point in the form $\begin{bmatrix} b \\ -a \\ 0 \end{bmatrix}$

If we convert into cartesian space, then we get

$$L_1 \times L_2 = \begin{bmatrix} bc_1 - cb / 0 \\ ca - ac_1 / 0 \\ 1 \end{bmatrix}$$ which is two parallel lines

intersect at (∞, ∞)

1.6) if L passes through x_1 then line equation is given by

$$L^T \cdot x_1 = 0$$

Similarly if L passes through x_2 then line equation is

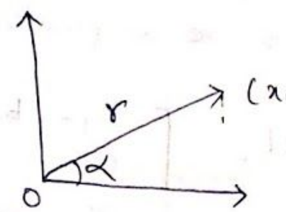
$$L^T x_2 = 0$$

This means L is a vector which is both perpendicular to x_1 and x_2. which means $L = x_1 \times x_2$

This is justified by the nature of cross product. i.e. cross product of two vectors is given by a third vector that is perpendicular to both two vectors

problem 2 :-

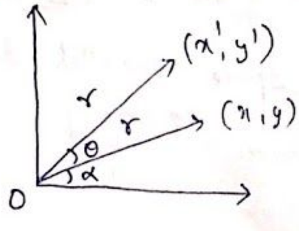
2.1)



Consider point (x, y) in 2D plane this can be written in polar coordinates in the form of r and α

i.e. $x = r\cos\alpha$, $y = r\sin\alpha$ — (1)

After rotation θ, let's assume new point is (x', y')



Now $\cancel{x' = r\cos(\alpha+\theta_1)}$

$$x' = r\cos(\theta + \alpha)$$
$$y' = r\sin(\theta + \alpha)$$

$x' = r\left[\cos\theta\cos\alpha - \sin\theta\sin\alpha\right]$, $y' = r\left[\sin\theta\cos\alpha + \cos\theta\sin\alpha\right]$

$x' = r\cos\alpha(\cos\theta) - (r\sin\alpha)\sin\theta$, $y' = r\cos\alpha(\sin\theta) + r\sin\alpha(\cos\theta)$

From(1) $x' = x\cos\theta - y\sin\theta$, $y' = x\sin\theta + y\cos\theta$

This can be written is matrix form as

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \quad — (1)$$

→ This is w.r.t to origin (0,0) the rotation θ. To rotate around a point (a,b) we first translate origin to (a,b) perform rotation and translate back to origin.

* when we move origin to (a, b) the (x,y) Changes to (x-a, y-b)

This is translation $(x,y)'$ to $(-a,-b)$

i.e.
$$x' = x - a$$
$$y' = y - b$$

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -a \\ 0 & 1 & -b \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \quad — (2)$$

* To move back to origin we translate (x, y) to (a, b)

i.e.
$$x' = x + a$$
$$y' = y + b$$

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \quad — (3)$$

Cascading (1), (2) and (3) in the order we get

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix}}_{\text{back to origin}} \underbrace{\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{rotation}} \underbrace{\begin{bmatrix} 1 & 0 & -a \\ 0 & 1 & -b \\ 0 & 0 & 1 \end{bmatrix}}_{\text{moving origin}} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Multiplying matrices from right to left

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & -a\cos\theta + b\sin\theta \\ \sin\theta & \cos\theta & -a\sin\theta - b\cos\theta \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} \cos\theta & -\sin\theta & -a\cos\theta + b\sin\theta + a \\ \sin\theta & \cos\theta & -a\sin\theta - b\cos\theta + b \\ 0 & 0 & 1 \end{bmatrix}}_{\text{final single } 3\times 3 \text{ homogenous matrix}} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

2.2) $P_1 = (1,1)$ $P_2 = (2,1)$ $P_3 = (2,2)$ $P_4 = (1,2)$

This can be solved using matrix obtained in the previous problem. First translate origin to P_2 and rotate 45° and translate back to origin.

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos 45^\circ & -\sin 45^\circ & -2\cos 45^\circ + 1\sin 45^\circ + 2 \\ \sin 45^\circ & \cos 45^\circ & -2\sin 45^\circ - 1\cos 45^\circ + 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 2 - 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} & 1 - 3/\sqrt{2} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

let P_1', P_2', P_3', P_4' be the new vertices

$$P_1' = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 2 - 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} & 1 - 3/\sqrt{2} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 - \frac{1}{\sqrt{2}} \\ 1 - \frac{1}{\sqrt{2}} \\ 1 \end{bmatrix} = \begin{bmatrix} 1.29 \\ 0.29 \\ 1 \end{bmatrix}$$

$$P_2' = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 2 - 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} & 1 - 3/\sqrt{2} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{2}{\sqrt{2}} - \frac{1}{\sqrt{2}} + 2 - \frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{2}} + \frac{1}{\sqrt{2}} + 1 - \frac{3}{\sqrt{2}} \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

$$P_3' = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 2-1/\sqrt{2} \\ 1/\sqrt{2} & +1/\sqrt{2} & 1-3/\sqrt{2} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2-1/\sqrt{2} \\ 1+1/\sqrt{2} \\ 1 \end{bmatrix} = \begin{bmatrix} 1.29 \\ 1.71 \\ 1 \end{bmatrix}$$

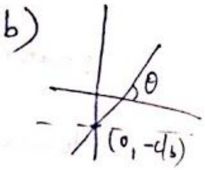
$$P_4' = \begin{bmatrix} \frac{1}{\sqrt{2}} & -1/\sqrt{2} & 2-1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} & 1-3/\sqrt{2} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2-2/\sqrt{2} \\ \frac{2}{\sqrt{2}}+\frac{1}{\sqrt{2}}+1-\frac{3}{\sqrt{2}} \\ 1 \end{bmatrix} = \begin{bmatrix} 0.59 \\ 1 \\ 1 \end{bmatrix}$$

Therefore the new vertices of the square are

$P_1' = (1.29, 0.29)$, $P_2' = (2, 1)$, $P_3' = (1.29, 1.71)$ $P_4' = (0.59, 1)$

Q.3) line $ax+by+c=0$

1. Translate origin to the point where $x=0 \Rightarrow (0, -c/b)$



$$x' = x - 0$$
$$y' = y - (-c/b)$$

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & c/b \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \quad — (1)$$

2. Rotate angle θ in clockwise i.e. rotate by $-\theta$

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \quad — (2)$$

3. Reflection along x-axis

$$x' = x$$
$$y' = -y$$

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \quad — (3)$$

4) Rotate angle back to θ

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \quad \text{—— } (4)$$

5) Translate back to origin from $(0, -c/b)$

$$x' = x + 0$$

$$y' = y + (-c/b)$$

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -c/b \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \quad \text{— } (5)$$

Cascading (1), (2), (3), (4), (5) in the order we get

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -c/b \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & c/b \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Multiplying all the homogenous matrices from right to left we get

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos^2\theta - \sin^2\theta & 2\sin\theta\cos\theta & \frac{2c}{b}\sin\theta\cos\theta \\ 2\sin\theta\cos\theta & \sin^2\theta - \cos^2\theta & -\frac{2c}{b}\cos^2\theta \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$\tan\theta = \frac{-a}{b} \Rightarrow \quad \sec^2\theta = 1 + \tan^2\theta \ \ldots \ b^2$$

$$\cos^2\theta = \frac{b^2}{a^2+b^2}$$

$$\sin\theta\cos\theta = \tan\theta\cos^2\theta \qquad \sin^2\theta = \frac{a^2}{a^2+b^2}$$

$$= \frac{-a}{b}\,\frac{b^2}{a^2+b^2}$$

$$= \frac{-ab}{a^2+b^2}$$

When $b = 0$

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & \frac{2c}{b} \times \frac{-ab}{a^2+b^2} \\ 0 & 1 & -\frac{2c}{b} \times \frac{b^2}{a^2+b^2} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & -\frac{2c}{a} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Problem 3:-

Given $x' = ax + by + t_x + \alpha x^2 + \beta y^2$

$$y' = cx + dy + t_y + \gamma x^2 + \theta y^2$$

This can be written in matrix form as

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \alpha & \beta & 0 & a & b & t_x \\ \gamma & \theta & 0 & c & d & t_y \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x^2 \\ y^2 \\ xy \\ x \\ y \\ 1 \end{bmatrix}$$

let h the matrix that needs to be computed to solve this

$$H = \begin{bmatrix} h_{11} & h_{12} & h_{13} & h_{14} & h_{15} & h_{16} \\ h_{21} & h_{22} & h_{23} & h_{24} & h_{25} & h_{26} \\ h_{31} & h_{32} & h_{33} & h_{34} & h_{35} & h_{36} \end{bmatrix}$$

$x' = H \times X \Rightarrow \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = H \begin{bmatrix} x^2 \\ y^2 \\ xy \\ x \\ y \\ 1 \end{bmatrix}$

$(x, y) \rightarrow$ normal

$(p, q) \rightarrow$ Homogeneous

In homogenous

let $p = \frac{x'}{z'}$ $q = \frac{y'}{z'}$

$$p = \frac{h_{11}x^2 + h_{12}y^2 + h_{13}xy + h_{14}x + h_{15}y + h_{16}}{h_{31}x^2 + h_{32}y^2 + h_{33}xy + h_{34}x + h_{35}y + h_{36}}$$

$$q = \frac{h_{21}x^2 + h_{22}y^2 + h_{23}xy + h_{24}x + h_{25}y + h_{26}}{h_{31}x^2 + h_{32}y^2 + h_{33}xy + h_{34}x + h_{35}y + h_{36}}$$

We need to find a matrix such that $Ah = 0$ to solve this system

$$\text{where } A = \begin{pmatrix} \{a_{x1}, a_{y1}\} \\ \vdots \\ \{a_{xN}, a_{yN}\} \end{pmatrix} \begin{matrix} \{\text{point } 1\} \\ \{\text{point } 2\} \\ \vdots \\ \{\text{point } n\} \end{matrix}$$

$$h = (h_{11}, h_{12}, h_{13}, h_{14}, \ldots, h_{21}, h_{22}, \ldots, h_{31}, h_{32}, \ldots h_{36})^T$$

$$a_x = [-x^2, -y^2, -xy, -x, -y, -1, 0, 0, 0, 0, 0, 0, px^2, py^2, pxy, px, py, p]^T$$

$$a_y = [0, 0, 0, 0, 0, 0, -x^2, -y^2, -xy, -x, -y, -1, qx^2, qy^2, qxy, qx, qy, q]^T$$

In the above ~~eqn~~ vectors a_x and a_y are calculated using points (x, y) in normal and (p, q) in homogeneous, After that h is calculated using $a_x^T h = 0$

$a_y^T h = 0$

~~This gives~~ For each point we have 2 equations, so for n points we have $2n$ equations

To solve for the Homogeneous matrix h which has 17 unknows we need ~~atleast~~ 9 points.

In the given case ~~where~~ (question), Homogeneous matrix that needs to be solved

$$h = [\alpha, \beta, 0, a, b, t_x, \gamma, \theta, 0, c, d, t_y, 0, 0, 0, 0, 0, 1]^T$$

Here we have 10 unknows and Hence minimum 10 questions i.e. a minimum of 5 point pairs $[(x_i, y_i), (x_i', y_i')]$ are needed.

$$\begin{bmatrix} x_1 & \cdots & x_5 \\ y_1 & \cdots & y_5 \end{bmatrix} \begin{bmatrix} x_1' & \cdots & x_5' \\ y_1' & \cdots & y_5' \end{bmatrix}$$

Now that we got A and h we ~~be~~ need to ~~sto~~ solve for h such that

$$Ah = 0$$

This can be done using linear least squares. find SVD of matrix A. ~~This will give a~~

$$u, s, vh = svd(A)$$

vh is of size 18 x 18. pick up the last row 18 values reshape into ~~vector~~ matrix of size (6, 3) to get h matrix. Finally divide by h[2][5] to get 1 at the ~~last column~~ last column, row value.