Exercise 1 Sthavishtha Bhopalam Rajakumar Oct 04, 2017

1 Introduction

The primary objective of the current exercise is to get acquainted with the Lattice Boltzmann (LB) code in a one-dimensional scenario. This has been accomplished through the following tasks :

1. 1D Advection-Diffusion problem
   - Simulation of a steep Gaussian profile
   - Simulation of a hyperbolic tangent profile
2. 1D Navier-Stokes (N-S) problem in simulating a shock tube

2 Algorithmic Description

The main steps in implementing a simple LB code include [1]:

- Computing the moments : density and momentum flux (only for N-S), from the relation [2] :

$$\begin{aligned} \rho &= \sum_i f_i \\ \rho u &= \sum_i c_i\ f_i,\ Only\ for\ N-S\ problem \end{aligned} \tag{1}$$

  Where ρ, ρu, c and f denote the macroscopic density, momentum flux, particle velocity and population.

- Computing the equilibrium populations for a D1Q3 model [1] :

$$\begin{aligned} f_0^{eq} &= \frac{2\rho}{3}(2\ -\ \sqrt{1\ +\ Ma^2}) \\ f_+^{eq} &= \frac{\rho}{3}(\frac{uc - c_s^2}{2c_s^2}\ +\ \sqrt{1\ +\ Ma^2}) \\ f_-^{eq} &= \frac{\rho}{3}(\frac{-uc - c_s^2}{2c_s^2}\ +\ \sqrt{1\ +\ Ma^2}) \end{aligned} \tag{2}$$

  Where, the subscripts (0,+ and -) associated with the populations denote it for the stationary, right moving (along the horizontal direction) and left moving particles associated with the D1Q3 model [1]. Additionally, c_s denotes the speed of sound, taken to be $1/\sqrt{3}$ when c is taken as unity. And Ma signifies the Mach number, which is the ratio of macroscopic fluid velocity and the speed of sound.

- Collision function : Performed by initially computing the relaxation parameter, (which is a function of kinematic viscosity or diffusion coefficient, depending on the problem) with subsequent updation of the particle populations. This can be accomplished as follows [1]:

$$\begin{aligned} f_i^{n+1} &= f_i^n + 2\beta(f_i^{eq,n} - f_i^n) \\ And,\ \beta &= \frac{1}{2\tau + 1} \end{aligned} \tag{3}$$

[1]Note: In the equilibrium equation for N-S, the uc component is taken to be the dot product of u and c_i (along each lattice direction)

where, τ is the relaxation time, related to the kinematic viscosity (for N-S problem) and diffusion coefficient (for advection-diffusion problem) as follows :

$$\begin{aligned} D &= \tau c_s^2,\ for\ Advection - Diffusion\ problems \\ \nu &= \tau c_s^2,\ for\ N - S\ problems \end{aligned} \tag{4}$$

Where D and ν denote the diffusion constant and kinematic viscosity repsectively.

- Advection (or Propagation) : This step is attempted by swapping the populations in accordance with the particle velocity c_i.

$$f_i^n = f_i^n(x - c_i) \tag{5}$$

- Boundary conditions : Three kinds of boundary conditions (BC) have been implemented in the current exercise [1]:
  - Periodic BC - In these BC, the incoming and leaving population to the flow domain are equated. To explain in detail (at every time instant),

    $$\begin{aligned} f_+(0) &= f_+(n-1) \\ f_-(n-1) &= f_-(0) \end{aligned} \tag{6}$$

    where, n denotes the number of nodes.
  - Bounce-back BC - Conserves the population at the boundaries, causing perfect reflection of the particle velocities.

    $$\begin{aligned} f_+(0) &= f_-(0) \\ f_-(n-1) &= f_+(n-1) \end{aligned} \tag{7}$$

  - Dirichlet BC - These are implemented when the particle velocities (as in the current exercise) are mentioned at the boundaries. Additinally, the populations at the boundaries are replaced by corresponding equilibrium values.

    $$f_i(boundaries) = f_i^{eq}(boundaries) \tag{8}$$

3 Results

3.1 1D Linear Advection-Diffusion Equation

3.1.1 Steep Gaussian Profile

The conditions given for the following problem include :

- $\rho(x, 0) = 1 + \frac{1}{2}\exp -5000(\frac{x}{n} - \frac{1}{4})^2$
- $D = 5\ x\ 10^{-8}$
- $u = 0.1$
- Periodic BC
- n = 800
- Number of time steps = 4000

The density profile obtained for this problem has been plotted in Figure 1. The thin lines indicate the initial density variation.

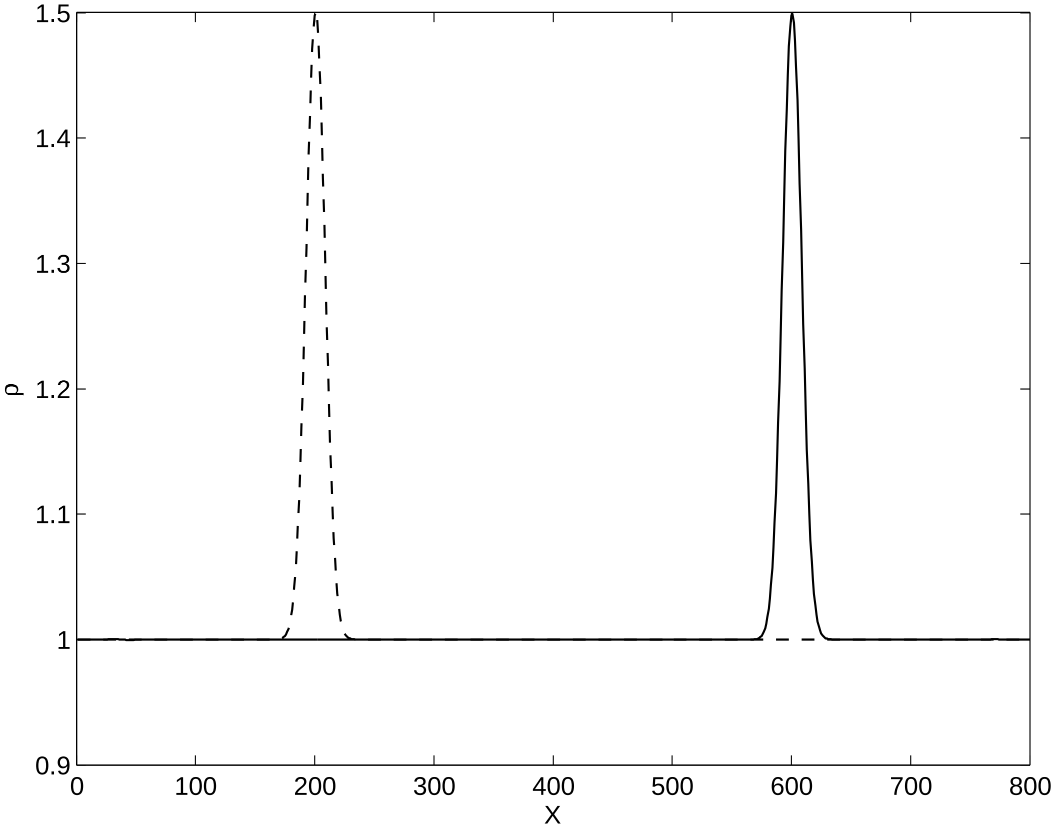

Figure 1: Density profile for a steep Gaussian profile

3.1.2 Hyperbolic Tangent Profile

The conditions given for the following problem include :

- $\rho(x,0) = 1 + \frac{1}{2}(1 - \tanh(\frac{x}{n} - \frac{2}{10})/\delta)$
- $D = 5$ x 10^{-8}
- $\delta = 0.01$
- $u = 0.1$
- Dirichlet BC, inlet and exit denisties equal to 1.5 and 1 respectively
- n = 800
- Number of time steps = 4000

The density profile obtained for this problem has been plotted in Figure 2. The thin lines indicate the initial density variation.

3.2 Navier Stokes Problem - Shock tube simulation

The conditions given for the following problem include :

- $\rho(x,0) = 1.5$ if $x \ < \ \frac{n}{2}$ and 1.0 otherwise
- $\nu = 0.06$
- $u(x,0) = 0$
- Bounce Back boundary condition

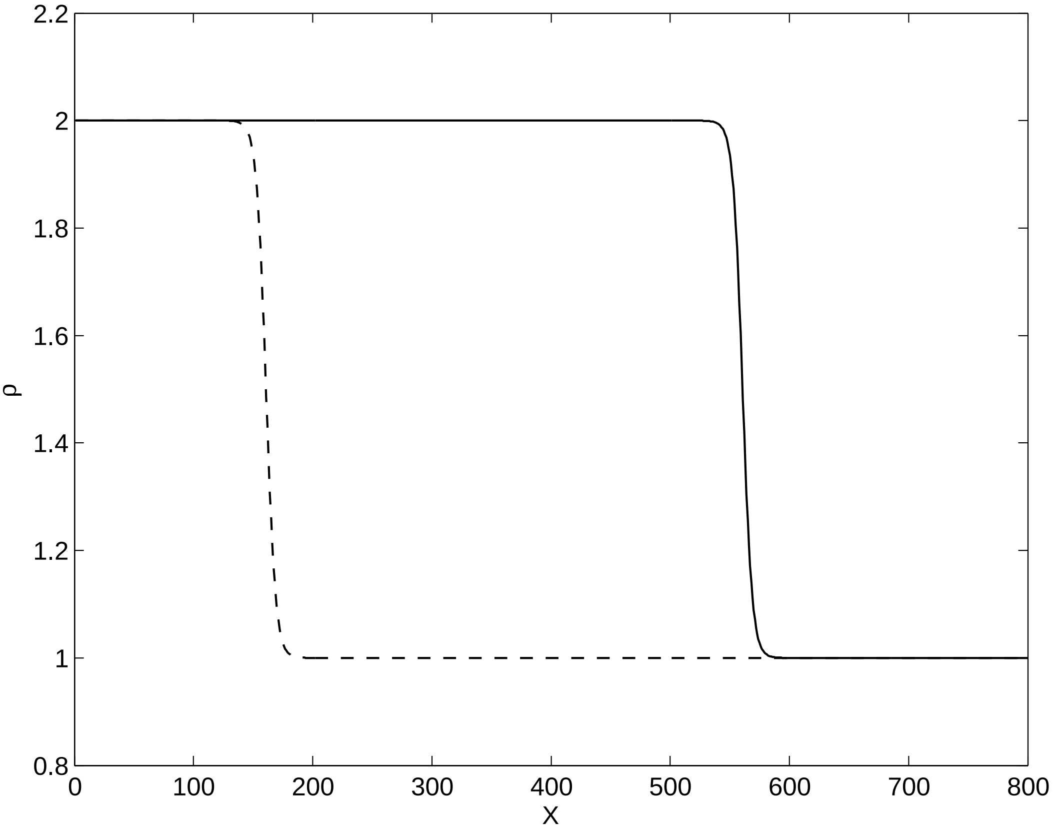

Figure 2: Density profile for a hyperbolic tangent profile

- n = 800
- Number of time steps = 500

The density profile obtained for this problem has been plotted in Figure 3.

4 Discussion

All the obtained results are consistent with the ones available in the literature [2, 1]. Thus, this verifies the credibility of our code.
Some interesting discussions about the effect of certain parameters : diffusion coefficient, macroscopic velocity, grid size, kinematic viscosity, Reynolds number and Mach number on the plots will be studied in this section.

4.1 1D Linear Advection-Diffusion Equation

4.1.1 Steep Gaussian Profile

- Effect of Diffusion constant D : For increasing values of D, the height of the gaussian profile decreased, with the subsequent increase in the diffusion of the fluid (and increase in relaxation time). However, the effect of decrease in D did not produce any significant change in the plot.
- Effect of macroscopic fluid velocity u : The density profile will shift forward or backward if u increases or decreases respectively.
- Effect of grid size n : Increase in n will only cause a rightward shift of the gaussian profile, with the same relative spacing between the peaks. A similar observation is evident for a decrease in n (except the case when the initial density is close to the left boundary) [2]

[2]Note that the above observation was made by keeping the number of iterations to be constant. Realistically speaking,

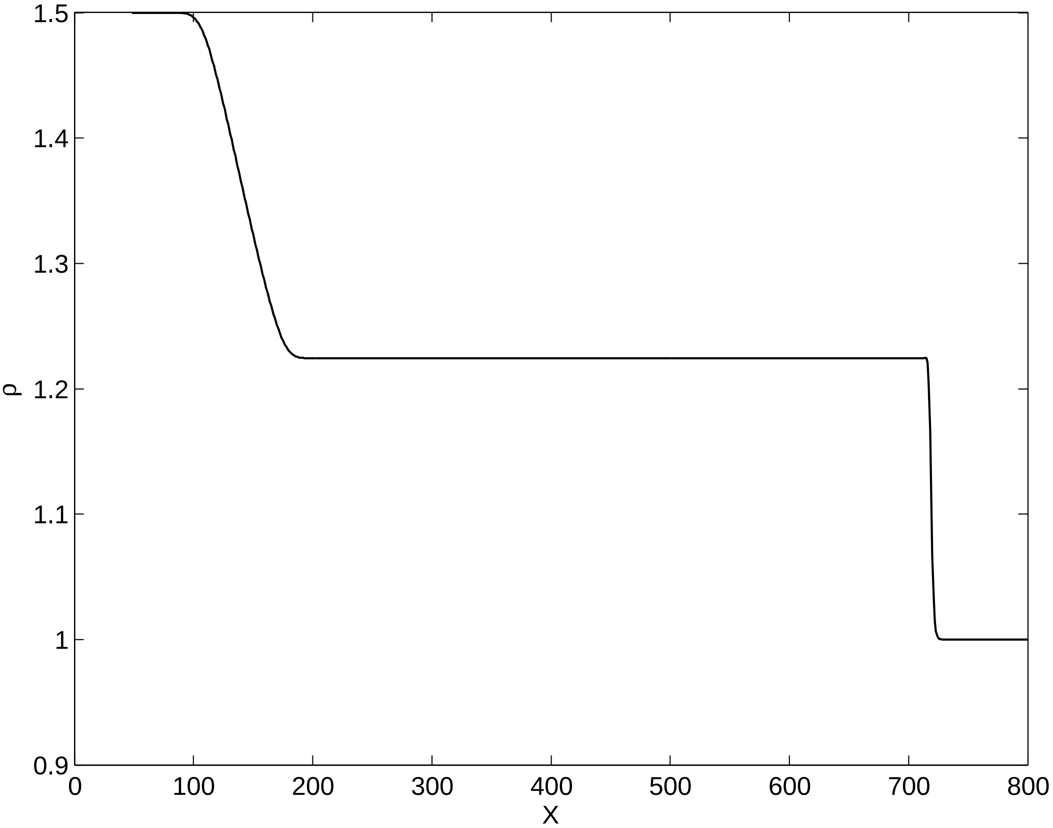

Figure 3: Density profile for a shock tube simulation

4.1.2 Hyperbolic Tangent Profile

- Effect of Diffusion constant D : Increase in D causes the tangent profile to smoothen at the top and bottom corners, due to subsequent diffusion. However, a decrease in D will cause the profile to become steep, tending towards a vertical line.
- Effect of macroscopic fluid velocity u : Same as explained for Steep Gaussian profile.
- Effect of grid size n : Same as explained for Steep Gaussian profile.

4.2 Navier Stokes Problem - Shock tube simulation

- Effect of kinematic viscosity ν : Decrease in ν induces spurious oscillations (due to the deficiency in the BGK model adopted in the current work [2]).
- Effect of grid size n : Same as explained for two previous cases.
- Maximum reported local Reynolds number : 1562.5
- Maximum reported local Mach number : 0.203

References

[1] Fabian Bösch and Ilya Karlin. Lattice boltzmann for the navier-stokes equations in 1d : Slides, 2017.

it would be wise to incorporate a convergence criteria in the code and capture the results accordingly. In that case, the density profile will be unaltered (a large number of iterations have to be run for convergence).

[2] Iliya V Karlin, Santosh Ansumali, Christos E Frouzakis, and Shyam Sunder Chikatamarla. Elements of the lattice boltzmann method i: Linear advection equation. *Commun. Comput. Phys*, 1(4):616–655, 2006.