CONTENTS

CHAPTER ONE

Innovation in the Mortgage Market

Susan Spring's determination had paid off. The Mortgage-Backed Securities (MBS) market did hold many wonderful opportunities. After her well-crafted presentation to her boss, Bill Stone, and the rest of the Fixed-Income Investment Strategy Committee, the group decided to make their initial foray into the MBS market. Susan was allowed to move 5% of the domestic fixed-income portfolio into MBS. She was told to keep the investments simple, basically PAC bonds and passthroughs.

The commitment proved to be successful since MBS did very well. The MBS sector outperformed the Treasury sector by nearly 100 basis points. Now, Susan was coming back to the Strategy Committee to request an increased exposure to the market. This time Susan was going for the big win; she was requesting that the allocation to MBS be upped to 30%, roughly its proportion of the domestic debt market. To get this proposal through, she was going to have to both convince her peers and then develop the tools to manage an even larger and more complicated portfolio. If this went well, Susan would be on her way to heading the Strategy Committee.

■ EXPECTED BACKGROUND OF READER

The reader of this book is assumed to have some exposure to financial markets, including the bond market. The basic concepts of finance and investing, including the structure of capital markets and discounted cashflows, should be familiar. All of the specialized mortgage terminology is defined in the workbook.

The book is intended for participants in the mortgage-backed securities market who are making investment decisions or aiding others in making investment decisions. Many of the tools described in the workbook are readily available from various vendors. This book serves to build a foundation for better use and understanding of these tools. The models described in the book have been simplified to facilitate the learning process. Investors should strive to build upon the knowledge gained here in understanding the more sophisticated analyses available.

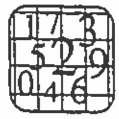
WORKBOOK ORGANIZATION

This workbook was designed to accompany Davidson and Herskovitz's *Mortgage-Backed Securities: Investment Analysis and Advanced Valuation Techniques* (Probus, 1994). The chapters of the workbook follow the outline of the text. Although it is not essential to an understanding of this workbook, we highly recommend that you read the text in conjunction with the workbook as you do the exercises. The MBS book contains more detail and a fuller explanation of the concepts contained in the workbook, as well as a description of the framework which the exercises complement.

The workbook provides a multitrack approach to developing the key concepts. In each chapter, there is a brief description of each concept with relevant definitions and equations. These concepts are reinforced through numerical examples. Then, two levels of exercises are provided.

The first level exercises are designed to form a basic understanding and can be done easily with just paper, a pencil, and perhaps a calculator. These exercises can be identified by the icon in the margin.

The second-level exercises are more complex and should be done on a spreadsheet, such as Microsoft Excel™ or Lotus 1-2-3,™ also indicated by the icon in the margin. These exercises will build upon one another, so in order to do these, one must go through the chapters of the workbook in consecutive order. Though the spreadsheet exercises are not necessary to grasp the fundamental concepts, they will enable the reader to have a fuller understanding of the complexities of mortgage valuation. The answers to the exercises are at the end of each chapter.



Lastly, we have included some questions for review as well as some issues to think about. These questions will *not* have written answers at the end of the chapter; they will be marked by the icon in the margin.

HISTORICAL OVERVIEW

Mortgages are a central part of American life. About two-thirds of the families in the United States own a home through mortgage financing. Mortgages were traditionally financed through deposits in financial institutions, primarily local thrift institutions such as savings and loan associations, that hold the mortgages they originate. The domination of mortgage originations by savings and loans (S&Ls) is shown in Figure 1-1. Life insurance companies contributed to mortgage capital by holding large portfolios of mortgages. Supplementing the thrift organizations, mortgage bankers developed the resale mortgage market by originating loans to sell to investors.

Figure 1-1[1]
S&Ls Dominate Mortgage Originations in 1983

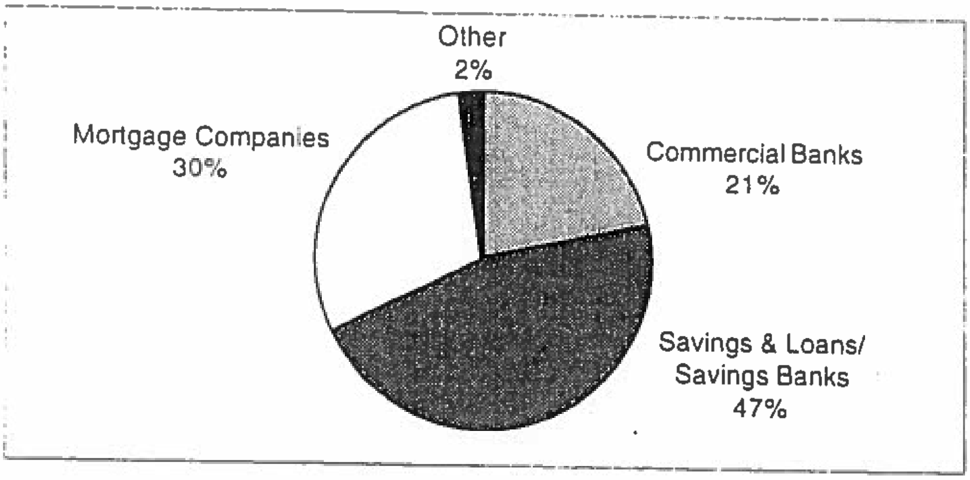
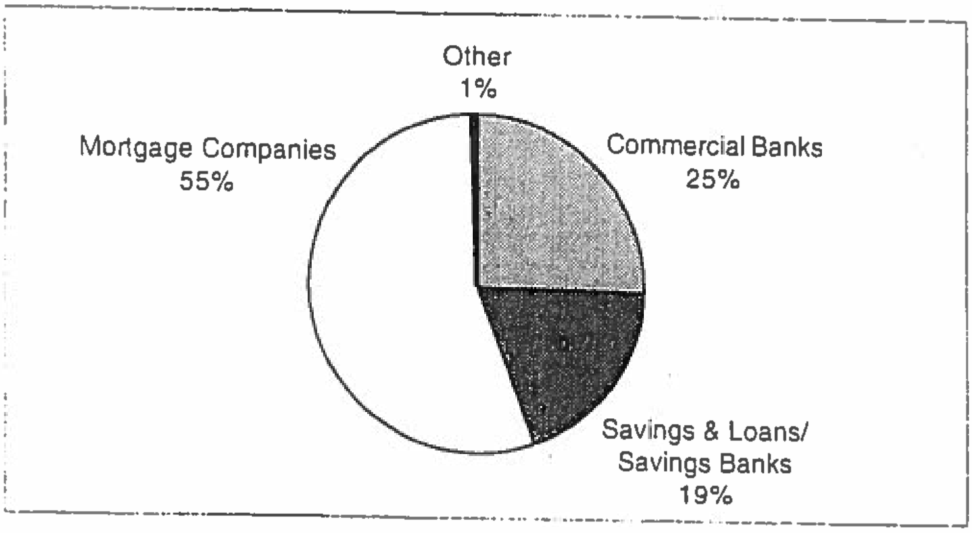


Figure 1-2
S&Ls' Market Share Falls to Less than 20% by 1995



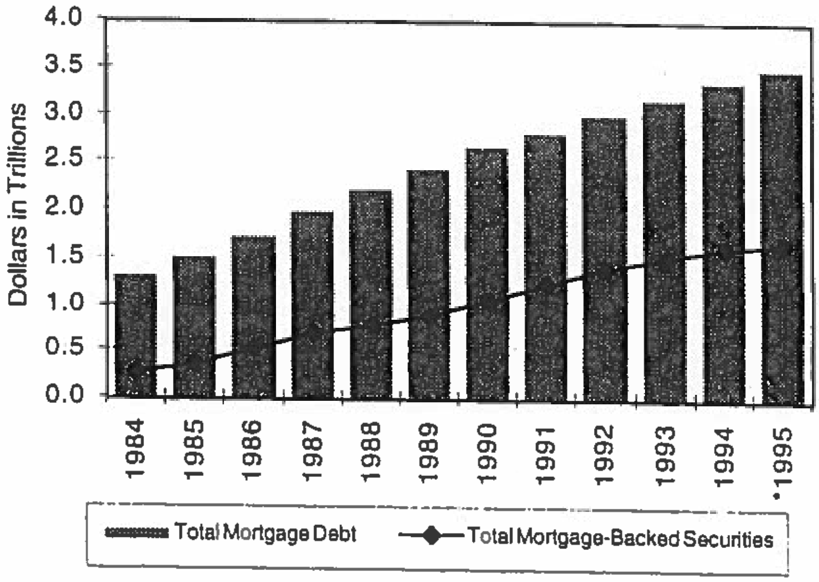
Jointly, the thrift institutions and mortgage bankers constituted the bulk of the mortgage market until the 1970s. At this time, interest

[1]Figures 1-1 through 1-5 are reprinted with permission of Inside Mortgage Finance, *The Mortgage Statistical Annual*, Copyright 1995, Bethesda, Maryland.

rates rose higher than the legal maximum on deposits, exposing the thrift institutions to disintermediation. The depositors diverted funds away from the thrift institutions into other higher-rate money-market investments. The thrifts' only recourse was to reduce lending and sell off mortgages. However, the high prevailing interest rates meant that mortgages had to be sold at a loss. This resulted in numerous institutions becoming financially insolvent and led to the S&L crisis of the 1980s. The thrifts' dilemma showed the need for an active secondary market where mortgages could be sold on a continual basis, rather than as an emergency measure. By 1995 the composition of the mortgage origination market had changed drastically, as shown in Figure 1-2. Mortgage companies that specialize in origination and servicing hold over 50% of the origination market. Commercial banks are rapidly gaining market share while the traditional role of the S&Ls has fallen to less than 20% of market share.

The growth of the secondary market in the 1970s was boosted by the Government National Mortgage Association (GNMA), the Federal Home Loan Mortgage Corporation (FHLMC), and Federal National Mortgage Association (FNMA). The three entities provided support for the mortgage market through the direct or implied credit of the United States, and created a standard for mortgage contracts through their mortgage purchasing programs.

Figure 1-3
The Growth of Mortgages and MBS

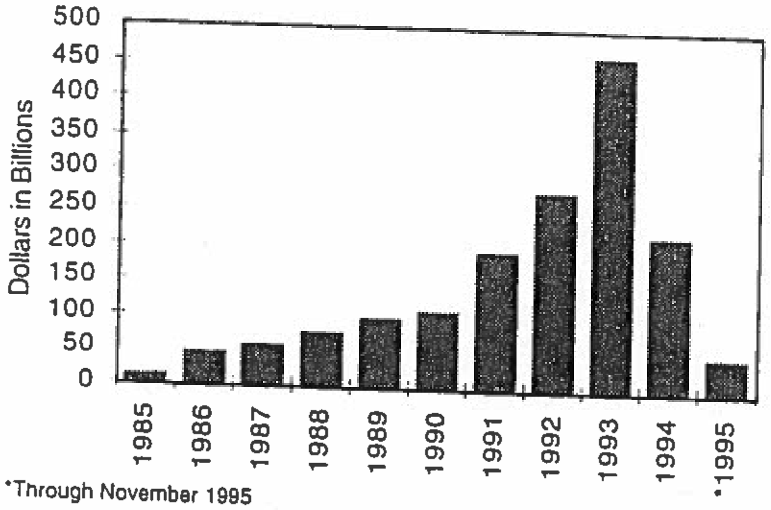


*First three quarters of 1995 only

The first mortgage-backed security (MBS) was introduced by GNMA in 1970. Since then, the MBS market has grown to over $1.6 trillion.

The growth of the MBS market is shown in Figure 1-3. Mortgage-backed securities created a demand for mortgages that increased the available funds for mortgages. The success of MBS can be attributed to its efficiency in reducing risk through diversification, and the sheer size of the market.

Figure 1-4
CMO Issuance



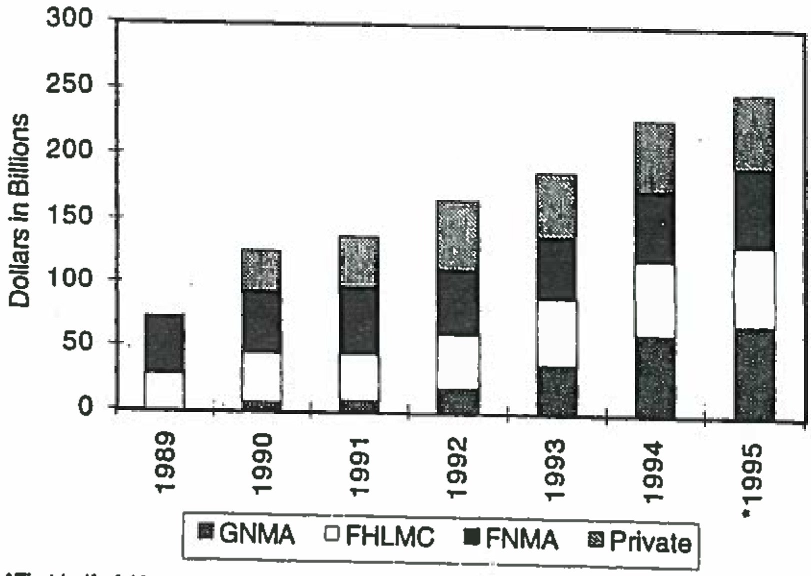
The first collateralized-mortgage obligation (CMO) was created in 1983. Wall Street took advantage of arbitrage opportunities in the huge mortgage market in an attempt to create value through carving up mortgage cashflows. The growth of the CMO market is shown in Figure 1-4. Wall Street enlarged the mortgage playing field by creating different types of securities from mortgage collateral. Instead of the traditional 30-year fixed-rate mortgage, investors could now pick from a seemingly unlimited range of bond characteristics. CMOs came in a variety of structures, maturity, and risk. In 1994, the dramatic turnaround in interest rates coupled with market uncertainty led to massive price drops in MBS. The negative publicity from a few well-publicized losses dramatically reduced CMO issuance. The diversity of the mortgage market is one of the reasons for its success but the complexity has taken many unwary investors by surprise.

Diversity in the mortgage market is not limited to CMO tranches. Originators competing for borrowers continually create new forms of mortgage contracts to meet the demands of increasingly financially savvy homeowners. The growth in adjustable-rate mortgage securities (ARMs), shown in Figure 1-5, reflects this trend.

The principles that have guided the mortgage-backed securities market have influenced the development of other asset-backed securities markets.

Credit cards, auto loans, home equity loans, manufactured housing, commercial mortgages, and even more exotic markets such as export receivables are all being securitized. The analytical tools described in this workbook will also provide insight into understanding these markets.

Figure 1-5
ARM Securities Outstanding



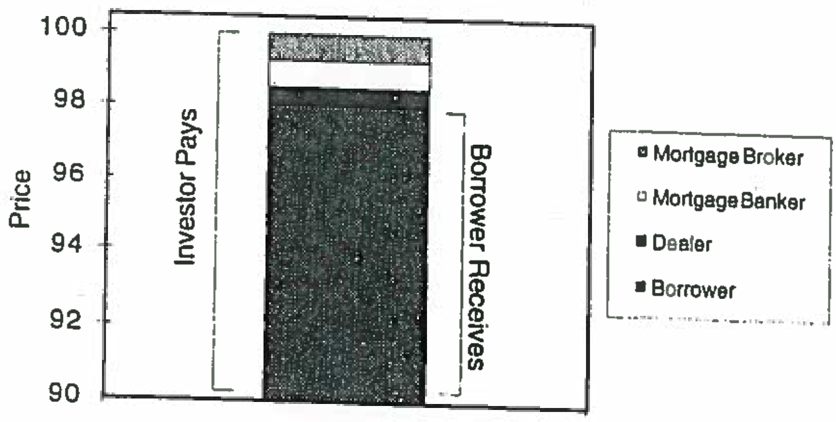
*First half of 1995 only

■ WHO GETS THE PIE?

The creation of a mortgage-backed security involves a number of players. First there is the borrower who wants a loan to buy a home. The borrower goes to the mortgage broker who arranges with a mortgage banker to originate the loan. The mortgage banker will securitize the loans. The security is sold to a dealer who will structure the MBS into a CMO or sell the MBS outright to a number of investors. Each of the intermediaries between the borrower and the final investor earns a portion of the proceeds. This breakdown is shown in Figure 1-6.

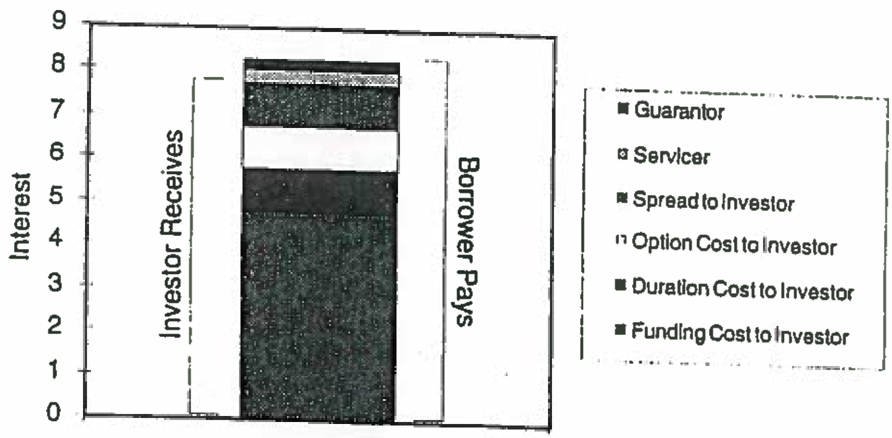
Once the loan has been originated, the proud homeowner will pay a monthly payment of principal and interest as determined by the loan contract. The loan servicer receives this payment and distributes it to the appropriate parties. The composition of the interest payment is shown in Figure 1-7. The servicer handles all of the paperwork for the loan and is in charge in the event of a default or prepayment. There are guarantees either at the loan or pool level by a government entity such as the Veteran's Administration (VA) or by a private guarantor. Both the servicer and guarantor will take as payment a percentage of the interest received from the borrower. The fees vary but are in the range of 20 to 75 basis points.

Figure 1-6
Who Gets the Price (Proceeds)?



The net interest, that is the gross interest payment from the borrower minus the servicing and guarantee fees, is received by the MBS investor. The net interest can be further broken down into four categories. The largest is the funding cost which is what the investor could get from investing in an alternative short-term risk-free instrument. The duration cost is the further yield the investor gets for investing in a long-term instrument since a standard mortgage has an average life ranging from 5 to 10 years. The option cost compensates the investor for the prepayment risk. And finally, the spread acts as a sweetener to induce the investor to take the risk of a MBS over a risk-free instrument such as a Treasury bond.

Figure 1-7
Who Gets the Interest?



■ CONCLUSION

Innovation is the byword of the mortgage market. From individual mortgage contracts to multilayered CMOs, the participants in the mortgage market continually try to create new value and compartmentalize risk to meet the changing expectations and requirements of investors. This workbook is designed to give the reader a basic understanding of the mortgage market as it is today, and to lay a groundwork of tools and skills with which to attack the challenges of tomorrow.

CHAPTER TWO

Overview of the Method

The meeting started out fine. Susan Spring presented her plans for increasing the size and scope of the MBS exposure. Bill Stone acknowledged that the MBS sector had done well but did not feel comfortable with such a large allocation to MBS. Some Committee members, perhaps fearful for their jobs, argued that MBS were too complicated. Susan indicated that the Domestic Fund was in the bottom quartile based on five-year performance. This was attributable to the lack of MBS exposure, she argued. This point carried the day and the Committee authorized a partial increase to the exposure; MBS could comprise up to 20% of the Domestic Fund. Susan was allowed to move into all aspects of the passthrough and CMO market. Not a complete victory, Susan thought, but a strong vote of confidence.

■ FRAMEWORK FOR ANALYSIS

This workbook presents a basic framework for evaluating mortgage-backed securities: environment, prepayments, cashflow, and analytics. The framework is outlined in Table 2-1. Building a model for MBS is akin to building a house: if the foundation is weak, regardless of how elaborate the house is on top, it probably will not outlive a big earthquake. Thus, the results of any analytics rest on the environmental assumptions. The environment drives the prepayment model, which defines the cashflows. Only when these steps have been done with care can useful results be obtained from the analytics. A good analyst understands the limitations of the foundation.

Environment

The first phase, environment, is the foundation of any modeling and analytics. The environment specifies the range and limitations of the final analysis. Factors left out in this phase will not be considered in the final analysis. The most influential environmental factor is the level of interest rates. The level of interest rates drives the homeowner's incentive to refinance and also affects the valuation of cashflows. Interest rate assumptions range from the very simple where rates are assumed to stay constant throughout time, to the very complex where rates are generated by a two-factor log-normal, mean-reverting process, which satisfies a no-arbitrage condition and whose variance and covariances are consistent with historical observation.

Table 2-1
Framework for Analysis

Phase of Analysis	Components	Discussed in Chapters
Environment	Interest Rate Economy Supply/Demand Regulatory/Tax/ Accounting	Chapter 4
Prepayments	Environmental Factors Modeling Approach	Chapter 5
Cashflow Structures	Mortgage MBS CMO/REMIC Portfolio	Chapter 6
Analysis	Valuation Risk Income	Chapters 3, 7, 8, 9, 10
Full Process of the Four Phases	Environment Prepayment Cashflow Analysis	Chapters 3, 7, 8, 9, 10

While interest rates are undeniably the most important, there are other factors that drive the cashflows of the underlying mortgages. We like to call MBS "warm-blooded securities" because MBS are influenced by all of the factors that influence homeowners. Any changes in the homeowner's life, whether it be income and employment or marriage and children, will affect borrower behavior. These types of individual decisions are difficult to quantify. However, since an MBS is a pool of many mortgages, the decisions of one borrower do not significantly impact the security. Analysts use economic data on employment and housing markets to predict the combined impact of these individual decisions.

Prepayments

Mortgages have a unique characteristic that distinguishes them from other fixed-income investments: prepayments. Under the standard mortgage contract, the borrower has the right to prepay a mortgage, either in part or in full, at any time during the life of the loan, usually without having to pay a penalty. This right means that the investor in a mortgage cannot know the maturity of the loan or the amount of total interest that will be received with certainty. It is this uncertainty that is the source of risk and reward in the mortgage market.

Quantifying prepayment risk is the never-ending task of Wall Street analysts. There are three basic reasons for prepayments: moving, refinancing, and default. Since most MBS have some sort of guarantee on principal, the investor realizes a default as a prepayment. The assumptions in the environment phase will drive the prepayments. Prepayments are usually quantified in two main forms: forecasts and models. A prepayment forecast is a specific numerical statement of future expected prepayment levels given a specific environment. A model, on the other hand, takes the form of mathematical equations that relate environmental assumptions to prepayment rates.

Cashflow

Cashflows form the basis for any analytics. All analytics attempt to summarize the characteristics of cashflow. Calculation of cashflow is perhaps the most straightforward phase analytically, but may be the most time consuming. Cashflow calculations are another building block process. The MBS cashflows are determined by the pool of mortgages. The CMO cashflows are based on the underlying MBS. Portfolio cashflows are based on the cashflows of the CMOs, MBS, and other securities in the portfolio.

The greatest complexity in this phase is in gathering the appropriate data that describes the mortgages. Each mortgage pool has unique characteristics that determine the cashflow. Another important consideration is the level of detail. Loan-level information is not available to investors for agency-guaranteed securities; so most analysis is based on weighted-average characteristics of the mortgage pools. As in all of the phases, there is a trade-off between additional accuracy and computational efficiency.

Analytics

The summary measures produced in the analytics phase provide insight into valuation, risk, and income. Valuation measures indicate the relative richness or cheapness of a security. Risk measures provide a guide to the range of outcomes for an investment. Finally, income measures indicate the pattern of possible future cashflows. These measures serve as a guide for comparison to other investment alternatives given investment objectives.

The accuracy and limits of these measures are bounded by the full analysis process. The performance indicated by the measures is a direct function of the assumptions in the environment and prepayment phases that are reflected in the cashflows and summarized in the analytics. If a static environment is assumed, the analytics produced will not inform the investor of the probable outcomes should interest rates move.

■ WHAT THE READER SHOULD HAVE LEARNED AFTER COMPLETING THE WORKBOOK

Chapter 3: Static analysis includes much of the basic MBS vocabulary and involves all four phases of analysis at the most simplistic level. The reader should gain a basic understanding of the workings of MBS and cashflows.

Chapter 4: This chapter lays the groundwork for more sophisticated environmental assumptions. Yield-curve calculations such as forward rates, the par-yield curve, and the zero-coupon curve are demonstrated. The reader will learn how to build a binomial tree and be introduced to option valuation.

Chapter 5: Prepayments and prepayment modeling are outlined. The reader will understand the elements involved in prepayment modeling. The analysis of prepayment data is stressed.

Chapter 6: MBS cashflows form the basis for understanding complex CMOs. The reader will create CMOs using a simplified approach that will show the main principles. A spreadsheet model can be built using the concepts developed in previous chapters.

Chapter 7: Relaxing the assumption of an unchanging environment and exploring the time dimension of returns are tackled in scenario analysis. Changing environments lead to analytical measures such as total return, effective duration, and convexity.

Chapter 8: This chapter combines concepts of earlier chapters and demonstrates how yield curve assumptions affect MBS cashflows and analytics. Common spread measures are introduced.

Chapter 9: The Option-Adjusted Spread (OAS) approach to valuation is discussed. The relationship between binomial and Monte Carlo analysis is explored, and a simple OAS model is constructed. OAS results are also discussed.

Chapter 10: Two common regulatory tests are demonstrated: FLUX and FFIEC tests. These measures involve variants of the scenario analysis method introduced in Chapter 7.

■ CONCLUSION

Running the most sophisticated OAS models and looking at the entire gamut of analytics are not a substitute for understanding one's own investment objectives. Only when investment objectives have been clearly defined, in terms of risk tolerance and cashflow needs, can an appropriate investment be selected. Your analytics may tell you that a 30-year FNMA 6.0 is selling at cheap levels, but if you are using this 30-year bond to support a debt that demands a floating rate linked to the three-month LIBOR, the FNMA cashflows would be a poor match to your liabilities.

Good MBS analysis requires matching the solution to the problem. In the following chapters, we will describe and show examples of a wide range of tools for analyzing MBS in our four-phase framework. We will show where and why the tools are effective and how they fall short. Open-ended questions will give the reader a feel for the complexities and issues in the MBS market that are beyond the scope of this workbook. Working through examples and concrete problems, we hope the reader will gain further insight into the methodology for choosing the right tools and learn the skills to develop tools according to the reader's unique requirements.

CHAPTER THREE

Applying the Framework: Static Analysis

Another case of Murphy's law in Susan's quest to get a higher exposure to MBS: an interesting bid list of seasoned MBS needed to be analyzed and Jerry Garcia had just died. Not that Sue really had any emotional ties to the Grateful Dead, but the intense retrieval of information about Garcia had brought down the Bloomberg system. This hadn't happened since John Wayne Bobbit. To get the bonds analyzed now would mean only one thing—setting up a simple calculator using a spreadsheet. Good thing Susan had spent the time in the past to actually derive MBS cashflows and build some simple tools of her own.

■ INTRODUCTION

This chapter develops key building blocks in analyzing MBS. In particular, we cover prepayment rates, mortgage cashflows, balance, yield, and static risk measures. Development of the fundamentals in this section will be necessary for the more advanced analysis later in the workbook. At the conclusion of this chapter, you should have a firm grounding in the fundamentals of mortgage math.

■ PREPAYMENT CONVENTIONS

Borrowers can return principal early by prepaying their loans fully or in part. Measuring the prepayments from an entire pool, which is a collection of loans, could be done by actually tabulating the prepaid dollars. This would be an accurate measure but an unwieldy way to compare prepayments across different pools.

The market has settled on expressing prepayments relative to the dollar balance of the pool. Three measures are currently used to report prepayment rates, shown in Table 3-1. Each measure has usefulness depending upon the type of security and context of the need to measure and compare rates.

Table 3-1
Prepayment Conventions

Single Monthly Mortality (SMM)	The SMM measures the percentage of dollars prepaid in any month, expressed as a percentage of the expected mortgage balance.
Conditional Prepayment Rate (CPR)	CPR reflects the percentage prepayment rate resulting from converting the SMM to an annual rate. The CPR is best understood as the percentage of the nonamortized balance prepaid on an annual basis.
Public Securities Association Model (PSA)	An industry convention adopted by the Public Securities Association in which prepayment rates, expressed in CPR, are assumed to follow a standard path over time. This path assumes that the prepayment rate for a pool of loans increases gradually over the first 30 months and then levels out at a constant rate. Along the 100% PSA curve, the prepayment rate starts at 0.2% CPR in the first month and then rises 0.2% CPR per month until month 30 when the prepayment rate levels out at 6% CPR.

In order to develop a familiarity with the conventions, we will present a few equations and then work through some numerical examples.

Equation 3-1
Single Monthly[1] Mortality (SMM)

$$\text{SMM} = 100 \times \frac{(\text{Scheduled Balance} - \text{Actual Balance})}{\text{Scheduled Balance}}$$

In Equation 3-1, the scheduled balance represents the expected balance based upon normal amortization.

Equation 3-2
Conditional Prepayment Rate (CPR)

$$\text{CPR} = 100 \times \left(1 - \left(1 - \frac{\text{SMM}}{100}\right)^{12}\right)$$

Equation 3-3
Public Securities Association Prepayment Model (PSA)

$$\text{PSA} = 100 \times \frac{\text{CPR}}{\text{minimum(age, 30)} \times 0.2}$$

Using the above equations, we can develop some examples of calibrating prepayment rates.

[1]A variant of the formula for calculating SMM, CPR, and PSA can be found in the PSA Standard Formulas manual. Many of the other formulas in this chapter can also be found there.

Calculating Prepayment Rates[2]

Using information about the scheduled balance, actual balance and age, we can calculate the prepayment rates in SMM, CPR, and PSA formats as seen below.

Table 3-2

Sample Prepayment Rate Calculations

Scheduled balance	154,000.00
Actual balance	153,000.00
Age (months)	25
SMM	0.65%
CPR	7.53%
PSA	150.54%

Exercise 3-1

Given an MBS with an age of three and a CPR of 1%, what is the equivalent PSA rate ?

$$PSA = 100 \times \frac{1\%}{3 \times .02} = \frac{1}{.06} = \cancel{16.7}$$

$$100 \times \frac{1}{.6} = 166.7$$

Exercise 3-2

Now repeat the exercise for the following pairs of ages and CPRs. The first month has already been calculated.

Work area

CPR	Age	PSA
1	1	500
1	2	250
1	3	~~83.3~~ 166.67
1	4	~~62.5~~ 125
1	5	~~50~~ .100
6	30	100
12	60	~~100~~ 200

$\frac{1}{2}(500) = 250.$

$\frac{250}{3} =$

$\frac{6}{30}(500) = \frac{1}{5}(500) =$

[2]SMM, CPR, and PSA all represent percentages. It is common, however, for the % sign to be dropped and to refer, for example, to 200% PSA simply as 200 PSA.

Exercise 3-3

Convert PSA rates to CPR.

Work area

$CPR = \frac{PSA \cdot \min(age, 30) \times .2}{100}$

PSA	Age	CPR
100	1	~~.2~~ .2
500	1	1
1000	1	2 2
1666	30	99.96
200	2	.8

$= 1 \times .2$

$\frac{1666 \times 30 \times .2}{100} =$

$\frac{200}{100} \times 2 \times .2$

$2 \times .4$

Exercise 3-4

Convert SMM to CPR and PSA equivalents.

Work area

Age	SMM	CPR	PSA
5	0.6	7.0	697
6	1.0	11.4	947
7	2.0	21.5	1538

■ CASHFLOWS

Mortgage payments made by fixed-rate borrowers have two interesting features: constant amounts and payment in arrears. A standard mortgage loan has a feature which allows the borrower to make a level payment each month. Part of this payment is allocated to interest and the remainder is used to pay down the principal. Initially the amount paid to interest greatly exceeds the amount allocated to principal. Over time this relationship changes and the allocation to principal increases greatly. The amount of the monthly cashflow is determined by using an annuity formula.

Payment in arrears means that the borrower makes the first payment one month after the loan is taken out. That is, a borrower who receives his loan on the first day of September makes the first payment on October 1. The interest amount will be based on the balance as of September 1.

To determine the monthly cashflow we use equation 3-4:

$$\text{Payment} = \frac{\text{Balance} \times \text{Coupon}/1200}{1 - (1 + \text{Coupon}/1200)^{-\text{Remaining term in months}}}$$

Equation 3-4
Monthly Payment for a Fixed-Rate Level Payment Loan

Coupon would be stated in annual terms. Use 8 for an 8% annual rate.

Determining the Monthly Payment

Table 3-3 shows an example of the monthly payment for a 9% mortgage with a balance of $200,000.

Table 3-3
Mortgage Characteristics

Balance	200,000.00
Coupon	9.00
Term (months)	360
Payment	1,609.25

$\frac{366}{120} = \frac{6}{20} = \frac{3}{10} = .3$

Exercise 3-5

What is the monthly payment required for a 30-year 8.5% mortgage having an original balance of $150,000 ?

$1221.22

Exercise 3-6

Fill in the following matrix showing ranges of coupon and original balance. Assume a 30-year mortgage and solve for the constant monthly payment using Equation 3-4. A few cells have already been calculated.

360

Work area

x3 x4

Coupon	$50,000	$100,000	Balance $150,000	$200,000	$250,000
4%	238.71 <-x2				
6%	299.78				
8%	366.88				
10%	438.79				
12%	514.31				
14%	592.87				

By examining the matrix, we can see a way to judge the relative importance of coupon and balance. For a $50,000 loan, the improvement to the borrower when rates drop 200 basis points is on the order of $67 per month but for a $250,000 loan the improvement is closer to $335 per month.

Seeing the changes in cashflow as they vary by balance may shed a little light on the general differences between prepayment rates on GNMAs versus FNMA/FHLMC MBS. Generally we observe GNMA MBS to have lower prepayment rates than conventionals. This could be attributed to the majority of GNMA borrowers who have less financial means than the conventional borrowers. However, it may be more likely that the GNMAs, with their lower loan balances, may provide less of an absolute incentive to refinance than the higher balance conventionals.

■ BALANCE

After we know how to determine the monthly cashflows, there is another formula we could use to calculate the remaining balance of a loan. This formula will prove useful when calculating the scheduled balances that serve as input to the SMM formulas.

To determine the percent of the original balance for a given month (t) apply the following equation:

Equation 3-5
Percentage of Principal Balance Outstanding

$$\%\ \text{Balance}_t = 1 - \frac{(1 + \text{Coupon}/1200)^{\text{Age}} - 1}{(1 + \text{Coupon}/1200)^{\text{Original Term}} - 1}$$

In this equation, the age and original term are expressed in months. The coupon would be expressed as a whole number, for instance, 9. Age indicates the number of months the loan has existed. Without any partial prepayments, age plus remaining term equals the original term.

Calculating the Scheduled Balance

Table 3-4
Example of the Percent of Original Balance Calculation

Original Term	360
Age	40
Coupon	8.50
Percent of Original Balance	97.21%

If our original balance had been $150,000 the scheduled balance at the end of month 40 would be $145,815.

Exercise 3-7

What would be the percentage of original balance outstanding at the end of month 5 for a 30-year loan with a coupon of 9%?

Exercise 3-8

Compute the percentage of original balance outstanding for the following loans having the indicated coupons and ages. Assume that the original term is 360 months. The first cell has already been calculated.

Work area

	Coupon		
Age	6%	8%	10%
60	93.05%		
120			
180			
240			
300			
359			

■ MONTHLY AMORTIZATION

Using the tools created with the monthly payments and balance formulas, we have the necessary building blocks to actually project the cashflows for a mortgage loan. Using these cashflows we can break the payment down between principal and interest. The amortization analysis can be taken one step further to incorporate prepayments.

From Equation 3-4 we can solve for the monthly loan payment of a mortgage. This payment can then be broken down into principal and interest as shown in Formula 3-6.

$$\text{Monthly Payment} = \frac{\text{Coupon}}{1200} \times \text{Starting Principal} + \text{Amortized Principal}$$

Equation 3-6
Monthly Principal and Interest

To solve for the amortized principal we could follow two methods. First we could plug, that is make amortized principal the difference between the monthly payment and the interest payment. Alternatively, we could project out the scheduled principal balances using Equation 3-5. The difference between balances in any month would equate to the amount of principal amortized.

Allocating Monthly Cashflow between Interest and Principal

Using the cashflow from Table 3-3, we can allocate the first month's payment between interest and principal, as shown in Table 3-5.

Table 3-5

Allocation of Monthly Payment between Interest and Principal

Balance	200,000.00
Coupon	9.00
Term (months)	360
Payment	1,609.25
Interest	1,500.00
Principal	109.25

Exercise 3-9

Assume a 30-year loan with an 8.5% coupon and starting balance of $200,000. Break down the monthly payment between interest and principal and fill in the missing cells of the table. In the exercise, you could solve for the balance in either of the two manners described above. That is, you could decrease the balance by subtracting out the amortized principal or you could use the scheduled balance formula. Try using both methods. The first month has already been filled in.

Work area

Month	Balance	Principal	Interest
0	200,000.00		
1	199,878.84	121.16	1,416.67
2			
3			
4			
5			
6			

Pools versus a Single Loan

It is at this point that the distinction between a single loan and a pool of loans has to be considered. Up to now, we have described the mortgage math as it applies to a single loan. A pool of loans works in a similar manner. Generally only loans with similar characteristics will be pooled together, therefore, one can think intuitively of a pool as one huge loan. While this analogy works most of the time, it breaks down with the inclusion of prepayments.

Consider a standard level-payment fixed-rate mortgage with a 30-year maturity. If this loan has "prepayments," either one of two things is happening. Either the borrower is paying a little extra principal each month (also called curtailments), or the loan has been completely paid off through refinancing or selling the home (or through default).

Now consider a pool of the same type of mortgages. What happens to the pool as people prepay? If there are curtailments, the average maturity of the pool will decrease, but the scheduled payment will remain the same. However, if the prepayments are due to refinancing, selling the home, or default, then the average maturity of the pool will not be affected. (To see this, think of the average of the following set: {30, 30, 30}. If one of the loans is totally prepaid, the average of {30, 30} is still 30.) What will change is the monthly scheduled payment. Partial prepayments are generally a very small portion of prepayments.

When working on a pool level, the pool characteristics are calculated as averages of the underlying mortgages, as described in Table 3-6. Thus the coupon becomes the "WAC" and the maturity becomes the "WAM." As individual mortgages in a pool prepay over time, the WAC and WAM will change, corresponding to the actual make-up of the pool.

Table 3-6

Weighted Average MBS Characteristics

Weighted Average Coupon (WAC)	The mean of the gross coupon of the underlying mortgages that collateralize a security, weighted by the corresponding principal balances.
Weighted Average Loan Age (WALA)	The mean of the age of the underlying mortgages that collateralize a security.
Weighted Average Maturity (WAM)	The mean of the remaining term of the underlying mortgages that collateralize a security.

Note: WAC, WALA, and WAM are all calculated based upon the current principal balance.

When there are partial prepayments, the WAM will shorten, as we discussed above. But the WALA will not change. Thus, WALA plus WAM does not always equal the original term of the mortgages.

Because pool-level data is not readily available, analytical models generally use the most current weighted-average characteristics. Forecasting into the future, one cannot know which loans will prepay, and so the convention is to assume that a pool's mortgages are made up of identical mortgages whose coupons and maturity equal the WAC and the WAM, and that prepayments are always in full. This means that, unlike for a single loan, the scheduled payment for a pool has to be recalculated at every period. Thus, while it might be convenient to think of a pool as a single loan, one must keep in mind that this does not always work.

Including Prepayments

Building on the knowledge of monthly amortization, we are now ready to incorporate prepayment rates. Normally prepayment rates are specified as either a PSA or CPR equivalent. When working back to the cashflow level, a translation will have to be made to turn assumed PSA or CPR into an SMM.

Recall from Equation 3-1 that we can measure the SMM by examining the difference between the scheduled and the actual balance. We can then modify the formula to solve for the actual balance at the end of the first month given a scheduled balance and an SMM. Restating Equation 3-1 in these terms would provide the following:

Equation 3-7
Applying SMM to Calculate Actual Balance for Month 1

$$\text{Actual Balance}_1 = \text{Scheduled Balance}_1 \times \left(1 - \frac{\text{SMM}_1}{100}\right)$$

Calculating the Actual Balance with Prepayment Rates

Let's return to the cashflow example set up in Table 3-5, however now we will apply a 1% SMM rate (corresponding to 11.36% CPR).

Table 3-7
Including Prepayments

Balance	200,000.00
Coupon	9.00
Term (months)	360
Payment	1,609.25
Interest	1,500.00
Principal	109.25
SMM	1.00%
Scheduled Balance	199,890.75
Actual Balance	197,891.84
Prepaid Principal	1,998.91

3

Equation 3-7 is somewhat restrictive because it only considers the first month. We can generalize Equation 3-7 to give the actual balance when the SMM is held constant, as seen in Equation 3-8.

$$\text{Actual Balance}_t = \text{Scheduled Balance}_t \times \left[1 - \frac{\text{SMM}}{100}\right]^t$$

Equation 3-8
Applying SMM to Calculate Actual Balance for Constant SMM

For any constant prepayment rate, we can determine the actual balance by only knowing the scheduled balance and age of the loan (represented by the subscript t). Equation 3-8 can be taken one step further, generalizing for the case when the SMM is not constant each month. This can be seen in Equation 3-9.

$$\text{Actual Balance}_t = \text{Scheduled Balance}_t \times \prod_{t=1}^{Age} \left(1 - \frac{\text{SMM}_t}{100}\right)$$

Equation 3-9
Applying SMM to Calculate Actual Balance for Non-constant SMM

The mathematical symbol Π means that we take the product of the terms starting from t=1 and proceeding until the actual age of the mortgage in order to find the cumulative prepayment amount. Equations 3-8 and 3-9 can be used to save computing time when examining the cashflow pattern of a mortgage in the case of prepayments. The scheduled balance vector must only be calculated once and then scaled depending upon the level of prepayments. A more expensive (based on computing time) approach would be to recompute the scheduled balance by re-amortizing the payments based upon the prepayments.

Exercise 3-10

Restate Equations 3-2 and 3-3 in the following manner:

Solve for SMM given the CPR.

Solve for the CPR given the PSA.

Exercise 3-11

Assume a 30-year loan with a 9% coupon and an initial starting balance of $150,000. Compute the scheduled balance and the actual balance based upon a SMM rate of 1%. The actual balance for the first month has already been computed.

Work area

Month	Scheduled Balance	Actual Balance
0	150,000.00	
1		148,418.89
2		
3		
4		
5		

Exercise 3-12

Assuming the same mortgage characteristics, compute the actual balances assuming a 10% CPR. The first month has again been computed.

Work area

Month	Scheduled Balance	Actual Balance
0	150,000.00	
1		148,607.54
2		
3		
4		
5		

The difference between scheduled and actual balances for any given month equals the prepaid principal. We could now add this additional term to our monthly amortization analysis. Monthly cashflows can now be broken down between interest, amortized principal, and prepaid principal.

For any month we would determine the normal interest and amortized principal using the standard equation. The SMM would then be used to translate the percentage prepayment rate into an absolute level, measured in terms of dollars.

Calculating Cashflows with Prepayment Rates

We can now tie various concepts together in the calculation of monthly cashflows. The example will consider an MBS with a remaining term of 360 months, coupon of 7.5%, prepayment rate of 150% PSA, and a starting balance of $200,000. Table 3-8 contains the cashflows for the first six months.

Table 3-8

Cashflows for a 7.5% MBS at 150% PSA

Month	Balance	Interest	Principal Amortized	Prepaid	PSA	CPR	SMM
0	200,000.00						
1	199,801.54	1,250.00	148.43	50.03	150	0.3	0.025
2	199,552.12	1,248.76	149.32	100.10	150	0.6	0.050
3	199,251.77	1,247.20	150.18	150.17	150	0.9	0.075
4	198,900.56	1,245.32	151.00	200.20	150	1.2	0.101
5	198,498.61	1,243.13	151.79	250.16	150	1.5	0.126
6	198,046.06	1,240.62	152.55	300.00	150	1.8	0.151

The key part of the example is the changing monthly cashflow. When prepayments occur, we must re-amortize the balance. For example, based on the standard terms of the loan above, we would have a monthly cashflow of $1,398.43. This in fact corresponds to the sum of the amortized principal and interest for the first month. However, after the first prepayment we must determine the new level at which to reset the monthly cashflows. That is, we must calculate the monthly payment to fully amortize a loan of $199,801.54 with 359 months of remaining term. This turns out to be $1,398.08 (which again equals the interest and amortized principal for the second month). There are shortcuts to repeatedly using the amortization formula to determine the monthly cashflow. For example, we can back into the principal paid by looking at the change in monthly balance.

Exercise 3-13

Determining the Monthly Cashflow

Using the same security characteristics as in Table 3-8, calculate the monthly cashflows. However, now assume a prepayment rate of 300% PSA. The cashflows for the first month have already been filled in.

Work area

Month	Balance	Interest	Principal Amortized	Prepaid	PSA	CPR	SMM
0	200,000.00						
1	199,751.37	1,250.00	148.43	100.20	300	0.6	0.05
2							
3							
4							
5							
6							

■ ARMS

Borrowers can choose between taking out a loan with a fixed interest rate or one with an adjustable rate. The coupon rates for adjustable-rate mortgages (ARMs) adjust off of some specified index, including one-year Treasury rates, LIBOR, and the Cost of Funds Index. The most common cost of funds index, the 11th district, represents the average interest rate paid by savings institutions in that district for their deposits.

Due to the coupon that adjusts according to an index, the cashflows for an ARM are not constant over time. Payments may rise and fall depending on the level of the index. Other features govern the amount by which the coupon paid by the borrower can change. The most common characteristics of ARMS can be found in Table 3-9.

To determine the coupon paid by the borrower, we sum the index plus the gross margin. For example, consider an ARM indexed to the constant maturity one-year Treasury bill (usually termed one-year CMT) that had a gross margin of 175 basis points. If the one-year T-bill rate were 6.1%, the borrower would pay a coupon of 7.85%. If the servicing and guarantee fee summed to 65 basis points, the net margin paid to the investor would be 110 basis points.

The periodic collars define the maximum range that the borrower's coupon can move at the reset date. If we go back to our previous example and assume a 1% periodic rate cap, then the coupon could rise to 8.85% or fall to 6.65%. Smaller periodic caps work in the favor of the borrower when interest rates are rising.

Lifetime caps and floors denote the maximum or minimum coupon that a borrower could pay. Based upon the previous example, and assuming the maximum lifetime coupon is 500 basis points above the initial coupon, the highest rate a borrower could pay would be 12.85%.

Table 3-9
ARM Characteristics

Index Value	The ARM coupon will be based upon a market index. The most common index is the 1-year Treasury rate.
Initial Coupon	The initial gross coupon reflects the rate paid by the borrower until the first reset date. Because of competitive market pressures ARMs are sometimes originated with a below-market "teaser rate."
Gross Margin	The spread over the index paid by the borrower. This spread includes both servicing and guarantee fees or credit enhancement. The guarantee fee only exists for loans insured by a Government Sponsored Enterprise (GSE), such as FNMA or FHLMC and government agencies.
Net Margin	The spread paid to the investor after deducting the servicing and guarantee fees or credit enhancement.
Reset Frequency	Standard amount of time between coupon resets. Some ARMs have a monthly reset, but most Treasury ARMs reset on an annual basis. On the other hand, an ARM indexed to COFI could have the coupon reset each month.
Periodic Rate Collar	The maximum amount at which the coupon paid by the borrowers is allowed to change at each reset. Typical conventional 1-year Treasury ARMs have a 2% annual rate collar, GNMAs have a 1% periodic rate collar.
Gross Life Cap	The highest potential interest rate paid by the borrower. In most cases the gross life cap would be between 500 and 600 basis points above the initial coupon.
Gross Life Floor	Lowest potential interest rate paid by the borrower.

Depending upon competitive pressures, some originators of ARMs may give borrowers a loan with a below market initial coupon. These coupons are termed "teaser rates" as they act as an incentive for a borrower to choose an adjustable rate over a fixed-rate mortgage contract. For example, although the fully indexed rate may be 7.85%, an originator may set the coupon at 5% for the first 6 or 12 months. Because lifetime caps are generally set as a spread to the initial coupon, teaser rate ARMS may provide an additional benefit to the borrower of a low lifetime cap.

ARM Resets

Assume that a borrower takes out a new ARM loan with terms specified in Table 3-10. Assuming that the current one-year CMT was 5.5%, the fully indexed rate would be 7.5% (5.5 + 2.0). The investor, however, would only receive 3.35% (4.0 – 0.65), during the first payment period because of the teaser rate of 4%. The fully indexed net coupon that the investor would receive if there hadn't been a teaser rate is 6.85 (7.5 – 0.65).

Table 3-10
Sample ARM Terms

Feature	Rate
Index Value	One-year Constant Maturity Treasury (CMT)
Initial Coupon	4%
Gross Margin	200 basis points
Servicing	65 basis points
Reset Frequency	12 months
Periodic Collar	2%
Lifetime Cap	10%

Let's assume that, at the first reset, the one-year CMT rate remained constant at 5.5%. At this point, the coupon paid by the borrower would be reset to 6%. Although the fully indexed rate would still be 7.5%, the adjustment is restricted by the periodic collar of 2%.

Exercise 3-14

Assume an ARM with a gross margin of 175 basis points, 65 basis points of servicing/guarantee fees, an initial coupon of 5.1%, periodic cap of 1%. What will the coupon reset to at the following dates, assuming the listed one-year CMT rates? The reset for the first year has already been calculated.

Work area

Year	1-Year CMT	Coupon Gross	Coupon Net
0		5.10	4.45
1	8.20	6.10	5.45
2	5.00		
3	5.75		
4	4.00		

■ MARKET CONVENTIONS

Loan and pooling factors add further complications to MBS. In particular, investors need to consider the effects of delay, accrued interest, and delivery variance when analyzing investment opportunities and in designing trading/settlement systems. Special pricing conventions may also confuse the newcomer.

Price Conventions

MBS prices are usually quoted as a percentage of the face value of the bond and in 32nds. For example, if a price was quoted of 102-10, that would be equivalent to 102 and 10/32. Converted into decimal, 102-10 would equal 102.313%. To calculate the actual dollar amount that would be paid, multiply 1.02313 times the balance or face value of the bond.

In the bond market, a 32^{nd} is commonly called a "tick." A half of a 32^{nd} (or 64^{th}) is called a plus and is denoted by a + sign. For example, 102-10+ equals 102.328 in decimal. A "par" bond means the bond is selling at a price of 100. A price below 100 is termed a "discount" while a price above 100 is selling at a "premium."

Delay

Mortgage borrowers make their payments in arrears, with payments normally due the first of the month (although payments can generally be received by the servicer later than the first without incurring any late charge.) The time between the due date of the loan payment and the cash payment made to the investor is called the delay.

Delay allows servicers time to process an enormous number of cashflows that do not all arrive on the first of the month. However, the delay also gives the various intermediaries an opportunity to earn float income on the cash waiting to be disbursed to the investors, while investors lose potential income. Since timing is an important factor in present value, delay is accounted for by the investors in the price paid for an MBS.

Costs of Delay

Assuming that the current risk free interest rate is 6.25% (quoted on an actual/actual basis), each day of delay costs $171.23 for an investor expecting a $1 million cashflow.

Exercise 3-15

Calculating the Cost of Delay

Assume that an investor is expecting a $1 million cashflow that has been delayed for 15 days and the current risk free interest rate is 6.50% (actual/actual). What is the total opportunity cost to the investor of having the payment delayed?

Accrued Interest

The accrual of interest is essentially a zero-sum issue for the investor. At the time of settlement the investor must pay the seller interest earned on the investment from the beginning of the month to the settlement date. In return, the investor is entitled to the entire monthly coupon. There is a subtle twist, though: the seller receives his accrued interest rate payment as of the settlement date, while the investor can only get his payment on a delayed basis.

Interest accrues starting from the first of the month and assumes that MBS pay interest on a 30/360 basis. Each month is assumed to have 30 days so the daily interest rate can be found as shown in Equation 3-10:

Equation 3-10
Daily Interest Rate

$$\text{Daily Coupon} = \frac{\text{Annual Coupon}}{360}$$

Exercise 3-16

Assuming settlement on the 15th, what would be the accrued interest on an 8.5% MBS with principal balance of \$5 million?

Variance and Pools Per Million

Passthrough MBS trading occurs either on a pool specific basis or on a To-Be-Announced (TBA) basis. When trading specific pools, buyers and sellers agree on price and amount of original face value. The seller has no latitude regarding the delivery specifications.

However, in the TBA market, the Public Securities Association (PSA) sets standards regarding acceptable delivery rules. These trading practices are meant to facilitate trading. They also provide opportunities for incremental profits.

Based on current standards, the seller can modify the amount of principal delivered. The dollar variance between actual and expected delivery must be between plus or minus 1% of the original trade amount. Taking advantage of the variance rules based upon market conditions is an accepted trading practice, and many dealers have designed sophisticated technology to use the variance to their advantage.

Exercise 3-17

A commercial bank agrees to sell a dealer $5mm of GNMA 9s on a TBA basis at a price of $103-16 and an allowed variance of 1%. At the settlement date the price of the GNMA 9s has increased to $104-00. Will the commercial bank deliver $5mm worth of GNMA 9s? If not, how much will they deliver?

With the continual maturation of the MBS market, many pools now trade with relatively small balances. Each pool requires clerical support for tracking and verifying principal and interest payments. Assuming a fixed cost per pool, many investors prefer to hold as few small pools as possible. To prevent sellers from dumping many small balance pools onto purchasers, the PSA has set guidelines regarding the maximum number of pools deliverable for TBA trades. These limits depend upon both the coupon of the security and the size of the trades. For coupons below 11%, a seller may deliver up to three pools per million dollars and for coupons 11% and above sellers can deliver five pools per million. There are additional limitations regarding tail pieces. For example, if two pools fall within the variance, a third small pool cannot be delivered.

■ ANALYSIS

The analysis section considers two aspects of MBS that are relevant in a static cashflow setting: yield and average life. In later chapters, the analysis will be extended to consider states of the environment when we do not specify a single interest rate scenario. The analysis tools developed in this section can be created in a spreadsheet application.

Yield

Yield is the standard measure of value for fixed-income securities. It presents a quick method to state the benefit of purchasing and holding a security until the final cashflow. In the context of finance, yield represents the internal rate of return assuming an initial purchase price and a stream of cash inflows.

In Equation 3-11 the subscript T represents the time after settlement (calibrated in months and including the actual delay days). The yield expressed in the formula above is called a mortgage equivalent yield as it assumes cash compounding occurring on a monthly basis. The price includes accrued interest.

The standard price and yield formula relationship can be seen in Equation 3-11:

Equation 3-11
Price and Yield for an MBS

$$\text{Price} = \frac{\text{Cash Flow}_{T_1}}{(1+\text{Yield}/1200)^{T_1}} + \ldots + \frac{\text{Cash Flow}_{T_{WAM}}}{(1+\text{Yield}/1200)^{T_{WAM}}}$$

$$\text{Price} = \sum_{i=1}^{WAM} \frac{\text{Cash Flow}_{T_i}}{(1+\text{Yield}/1200)^{T_i}}$$

Computing the Yield

Assume an annual coupon paying security with a coupon of 5% and a current price of $101 and two years until maturity. The cashflows and yield have been computed in Table 3-11.

Table 3-11
Sample Yield Calculation

Year	Cashflow
0	−101.00
1	5.00
2	105.00
Yield	4.00%

Exercise 3-18
Using the bond from Table 3-11, what would the yield be if the initial price were $99? (Hint: This is done easily using a spreadsheet and an IRR or goal seek function. IRR functions are also available on some calculators. If you are doing it by hand, you will need to use the quadratic formula or a trial and error approach.)

why 5.5% should = 6%

Standardizing Yield

The yield calculation shown in Equation 3-11 assumes that the MBS pays on a monthly basis. As a market convention, most yields are stated on a semi-annual basis which is called bond equivalent yield. That is, yields are quoted as if a security pays a coupon twice a year. In order to compare the yield calculated in Equation 3-11 to other securities, we must adjust the compounding basis from a monthly yield to a semi-annual equivalent. Intuitively, we're trying to solve the following question: What is the semi-annual coupon rate at which the investor is indifferent between receiving monthly or semi-annual payments?

This question can be expressed in the following formula (MEY represents the yield as quoted in monthly terms while BEY represents a yield quoted based on semi-annual compounding):

$$\left(1+\frac{MEY}{1200}\right)^{12} = \left(1+\frac{BEY}{200}\right)^{2}$$

Equation 3-12
Monthly and Bond Equivalent Yields

Solving for the BEY as a function of the MEY:

$$BEY = 200 \times \left(\left(1+\frac{MEY}{1200}\right)^{6} - 1\right)$$

Exercise 3-19

Assuming that an MBS had a monthly equivalent yield of 8%, what would be the bond equivalent yield ?

Average Life

As a concept, maturity works well to describe the period that an investor in corporate or Treasury bonds keeps his principal outstanding. However, with MBS the maturity can be rather misleading. For a 30-year MBS, it is quite true that the final payment, or maturity, occurs with the 360th cashflow. However, the amortizing nature of the security provides a return of principal to the investor over the entire 30-year period.

In lieu of maturity, investors rely on the weighted average life (WAL) of the MBS as a proxy for maturity. The WAL indicates the average amount of time that $1 of principal remains outstanding. WAL can be calculated based on the following formula:

$$\text{Average Life} = \frac{\sum_{i=1}^{WAM} T_i \times \text{Principal}_i}{\sum_{i=1}^{WAM} \text{Principal}_i}$$

Equation 3-13
Weighted Average Life

In Equation 3-13 the variable T represents the time until the principal payment. This time dimension could be stated in months or days. Normally the WAL will be quoted in years, so the calculation must be adjusted to place the time into the proper units. The principal payments must also be adjusted for the delay.

Calculating Average Life

Assume a security that amortizes annually over five years, using the amortization schedule below the average life equals 3.5 years. The principal cashflows are shown in Table 3-12.

Table 3-12

Principal Cashflows

Time	Principal
1	10.00
2	15.00
3	20.00
4	25.00
5	30.00
Total Principal	100.00
WAL	3.50

Exercise 3-20

Given the amortization schedule in the table below compute the average life.

Work area

Time	Principal
1	5.00
2	10.00
3	20.00
4	40.00
5	80.00
Total Principal	155.00
WAL	

Cashflow Duration

Duration is a tricky issue in regard to MBS. Normally we think of duration as providing some measure for the time value weighting of cashflows. Duration can alternatively be used as a risk measure. In this case, duration would represent the price sensitivity to a change in interest rates. MBS market participants should be aware that duration can come in three basic varieties: Macaulay, modified, and effective.

Macaulay Duration

The Macaulay duration represents a time-weighted value of cashflows. For a security whose cashflows do not change with interest rates, the Macaulay duration can be shown to equal the percentage change in price for a percentage change in yield.

$$\text{Duration} = \frac{1}{\text{Price}} \times \left(\sum_{i=1}^{WAM} \frac{T_i \times \text{Cashflow}}{(1 + \text{yield}/200)^{2T_i}} \right)$$

Equation 3-14
Macaulay Duration

Where:

T = the time elapsed from settlement until receipt of cashflows

Yield = bond equivalent yield

Table 3-13 computes the Macaulay duration for a 2-year, semi-annual 6% coupon-bearing security, currently priced at $100 having a yield of 6%.

Table 3-13
Sample Macaulay Duration Calculation

Time	Cashflow	Time Weighted Discounted Cashflow
0.5	3.00	1.4563
1.0	3.00	2.8278
1.5	3.00	4.1181
2.0	103.00	183.0283
Yield	6.00	
Price	100.00	
Duration	1.91	

The third column in the table represents the term that is summed in Equation 3-14. This term has been labeled the time-weighted discounted cashflow.

Modified Duration

The modified duration makes a slight change in the Macaulay duration formula. It represents the percentage change in price for a basis point change in yield, rather than a percentage change in yield.

Equation 3-15
Modified Duration

$$\text{Modified Duration} = \frac{\text{Duration}}{1+\dfrac{\text{Yield}}{200}}$$

Exercise 3-21

Convert the Macaulay duration from Table 3-10 to a modified duration.

Effective Modified Duration

For a security like an MBS, measures of modified and Macaulay duration have little meaning when it comes to interest rate risk management. Both modified and Macaulay duration do not take into account the effect changes in interest rates have on MBS cashflows through prepayments. This cashflow sensitivity of MBS leads us to use an empirical calculation for duration shown in Equation 3-16, below.

The prices in the shifted scenarios will reflect changing prepayments and pricing assumptions. These prices can be estimated by either using a valuation model, such as OAS, or by determining an approximate change in the prepayment rates and spread of an MBS to an appropriate Treasury benchmark based on the shift in the yield curve.

Equation 3-16
Effective Modified Duration

$$\text{Effective Modified Duration} = \frac{-100}{\text{Price}_{\text{Base}}} \times \frac{\text{Price}_{+\Delta\text{Yield Scenario}} - \text{Price}_{-\Delta\text{Yield Scenario}}}{2 \times \Delta\text{Yield}}$$

Issues related to the effective duration will be explored further in Chapter 7.

Exercise 3-22 (Advanced)

Create a graph with age on the x-axis and CPR on the y-axis. Make plots of the curves representing 50, 100, 200 and 500% PSA.

Exercise 3-23 (Advanced)

Based on the analysis in Exercise 3-8, compute the percentage of scheduled balances for the following table.

Work area

8% Coupon

Age	180-Month Original Term	360-Month Original Term
24		
48		
72		
96		
120		
144		
168		

Should the scheduled balance for the 360-month original term loan always be twice the amount as the scheduled balance for the 180-month loan?

Exercise 3-24 (Advanced)

Assume a newly created 30-year, 8% MBS. Break down payments over the first six months between interest, amortized principal, and prepayments. First, assume a prepayment rate of 7% CPR. In the second case, assume that prepayments will occur according to 300% of the PSA model. Using a spreadsheet, extend the results to the maturity of the MBS.

Work area

7% CPR

Month	Monthly CPR	Actual Balance	Interest	Principal Scheduled	Principal Prepaid
0		150,000.00			
1	7%				
2	7%				
3	7%				
4	7%				
5	7%				
6	7%				

300% PSA

Month	Monthly CPR	Actual Balance	Interest	Principal Scheduled	Principal Prepaid
0		150,000.00			
1					
2					
3					
4					
5					
6					

■ REVIEW QUESTIONS

Why do borrowers choose ARMs instead of fixed-rate loans? What might the implications be regarding expected prepayment effects?

Is it more reasonable to determine the effect of delay by using the risk-free rate or by assuming another discounting rate?

Why should the bond equivalent yield be higher than the monthly equivalent yield? Why is CPR less than 12 times the SMM?

ANSWERS TO EXERCISES

3-1

$$\text{PSA} = 100 \times \frac{1}{\text{minimum}(3,\ 30) \times 0.2}$$

$$\begin{aligned}\text{PSA} &= 100 \times \frac{1}{3 \times 0.2} \\ &= 166.67\end{aligned}$$

3-2

CPR	Age	PSA
1	1	500
1	2	250
1	3	167
1	4	125
1	5	100
6	30	100
12	60	200

3-3

PSA	Age	CPR
100	1	0.20
500	1	1.00
1,000	1	2.00
1,666	30	99.96
200	2	0.80

3-4

Age	SMM	CPR	PSA
5	0.6	6.97	697
6	1.0	11.36	947
7	2.0	21.53	1,538

3-5

Monthly payment is $1,153.37.

3-6

Coupon	$50,000	$100,000	Balance $150,000	$200,000	$250,000
4%	238.71	477.42	716.12	954.83	1,193.54
6%	299.78	599.55	899.33	1,199.10	1,498.88
8%	366.88	733.76	1,100.65	1,467.53	1,834.41
10%	438.79	877.57	1,316.36	1,755.14	2,193.93
12%	514.31	1,028.61	1,542.92	2,057.23	2,571.53
14%	592.44	1,184.87	1,777.31	2,369.74	2,962.18

3-7

Percentage of original balance outstanding is 99.7228%.

3-8

Age	Coupon 6%	8%	10%
60	93.05%	95.07%	96.57%
120	83.69%	87.72%	90.94%
180	71.05%	76.78%	81.66%
240	54.00%	60.48%	66.41%
300	31.01%	36.19%	41.30%
359	0.60%	0.73%	0.87%

3-9

Month	Balance	Principal	Interest
0	200,000.00		
1	199,878.84	121.16	1,416.67
2	199,756.82	122.02	1,415.81
3	199,633.94	122.88	1,414.94
4	199,510.19	123.75	1,414.07
5	199,385.56	124.63	1,413.20
6	199,260.04	125.51	1,412.31

3-10

$$SMM = 100 \times \left(1 - \left(1 - \frac{CPR}{100}\right)^{\frac{1}{12}}\right)$$

$$CPR = \frac{PSA \times \min(\text{age}, 30) \times 0.2}{100}$$

3-11

Month	Scheduled Balance	Actual Balance
0	150,000.00	
1	149,918.07	148,418.89
2	149,835.52	146,853.79
3	149,752.35	145,304.56
4	149,668.56	143,771.02
5	149,584.14	142,253.03

3-12

Month	Scheduled Balance	Actual Balance
0	150,000.00	
1	149,918.07	148,607.54
2	149,835.52	147,227.36
3	149,752.35	145,859.35
4	149,668.56	144,503.40
5	149,584.14	143,159.42

3-13

Month	Balance	Interest	Principal Amortized	Prepaid	PSA	CPR	SMM
0	200,000.00						
1	199,751.37	1,250.00	148.43	100.20	300	0.6	0.050
2	199,401.38	1,248.45	149.28	200.71	300	1.2	0.101
3	198,949.94	1,246.26	150.06	301.37	300	1.8	0.151
4	198,397.13	1,243.44	150.77	402.04	300	2.4	0.202
5	197,743.16	1,239.98	151.41	502.56	300	3.0	0.254
6	196,988.40	1,235.89	151.97	602.79	300	3.6	0.305

3-11

Month	Scheduled Balance	Actual Balance
0	150,000.00	
1	149,918.07	148,418.89
2	149,835.52	146,853.79
3	149,752.35	145,304.56
4	149,668.56	143,771.02
5	149,584.14	142,253.03

3-12

Month	Scheduled Balance	Actual Balance
0	150,000.00	
1	149,918.07	148,607.54
2	149,835.52	147,227.36
3	149,752.35	145,859.35
4	149,668.56	144,503.40
5	149,584.14	143,159.42

3-13

Month	Balance	Interest	Principal Amortized	Prepaid	PSA	CPR	SMM
0	200,000.00						
1	199,751.37	1,250.00	148.43	100.20	300	0.6	0.050
2	199,401.38	1,248.45	149.28	200.71	300	1.2	0.101
3	198,949.94	1,246.26	150.06	301.37	300	1.8	0.151
4	198,397.13	1,243.44	150.77	402.04	300	2.4	0.202
5	197,743.16	1,239.98	151.41	502.56	300	3.0	0.254
6	196,988.40	1,235.89	151.97	602.79	300	3.6	0.305

3-14

Year	CMT1	Coupon Gross	Coupon Net
0		5.10	4.45
1	8.20	6.10	5.45
2	5.00	6.75	6.10
3	5.75	7.50	6.85
4	4.00	6.50	5.85

3-15
Delay cost is $2,671.23.

3-16
Accrued interest is $16,527.78. (Note: Settlement is on the 15th, so there are 14 accrual days.)

3-17
The bank will not deliver $5 million; it will deliver $4.95 million.

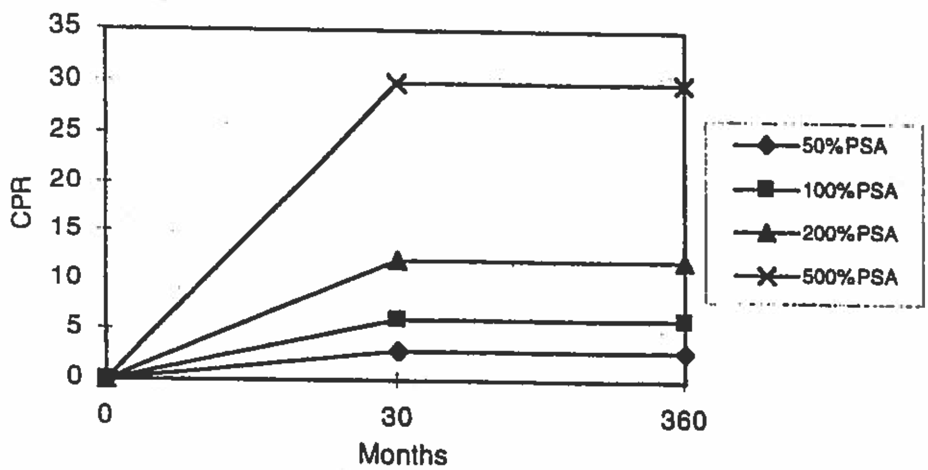
3-18
The yield is 6%.

3-19
The bond equivalent yield is 8.1345%.

3-20
The WAL is 4.16129 years.

3-21
The modified duration is 1.85.

3-22



3-23

8% Coupon

Age	180-Month Original Term	360-Month Original Term
24	92.51%	98.26%
48	83.72	96.22
72	73.41	93.83
96	61.31	91.02
120	47.13	87.72
144	30.50	83.86
168	10.99	79.33

No, the scheduled balance for a 360 term loan is not twice the amount of a 180-month loan.

3-24

7% CPR

Month	Monthly CPR	Actual Balance	Interest	Principal Scheduled	Principal Prepaid
0		150,000.00			
1	7	148,995.56	1,000.00	100.65	903.79
2	7	147,997.12	993.30	100.71	897.73
3	7	147,004.64	986.65	100.77	891.71
4	7	146,018.09	980.03	100.83	885.73
5	7	145,037.42	973.45	100.89	879.78
6	7	144,062.61	966.92	100.95	873.87

300% PSA

Month	Monthly CPR	Actual Balance	Interest	Principal Scheduled	Principal Prepaid
0		150,000.00			
1	0.6	149,824.20	1,000.00	100.65	75.16
2	1.2	149,572.38	998.83	101.27	150.55
3	1.8	149,244.46	997.15	101.84	226.08
4	2.4	148,840.48	994.96	102.36	301.62
5	3.0	148,360.59	992.27	102.84	377.06
6	3.6	147,805.04	989.07	103.26	452.28

CHAPTER FOUR

Environmental Alternatives

Now that Susan was moving into more complicated aspects of the MBS market, new terms kept on cropping up. The research reports were constantly referring to things like forward rates, term structures of volatility, par curves, and interest rate trees. Susan had a basic grasp of these financial terms from her MBA training and CFA study courses, but her favorite salesman told Susan that he did not believe in forward rates. Susan was at a loss. There seemed to be another world going on below the surface of the analysis. Now was the time to get a cup of latté and become an honorary member of the rocket science club.

INTRODUCTION

While there are many factors that influence the behavior of mortgage-backed securities (MBS), the largest environmental factor is the level of interest rates. Interest rates play an important role in determining mortgage prepayments because the level of interest rates relative to a homeowner's mortgage rate creates the incentive to refinance. For MBS valuation, the rate at which cashflows are discounted is determined by a yield curve, usually the Treasury. Because future cashflows are uncertain, the level of interest rate volatility is crucial for building probable interest rate scenarios. Of course, there are many other variables that should be considered in MBS valuation such as housing market activity and overall economic growth but we will not discuss them here. For ease of calculation, in this chapter interest rates are often expressed in decimal form. For example, 5% equals 0.05.

FORWARD RATES

Forward rates can be thought of as the market's assessment of how it values future cashflows. MBS are virtually always priced from models that make forward rate assumptions. The calculation of forward rates from existing spot rates is a simple exercise.

Suppose you have to choose between two investments, A and B. Investment A is a two-year zero-coupon bond with a yield of 7.0%. Investment B is a series of two one-year zero-coupon bonds, the second bond bought after the first bond matures. The first one-year bond has a yield of 5%. The one-year forward rate would be the yield on the second bond, which makes the investor indifferent between the two investment strategies.

Using standard compounding math, we can find the return from the two investment strategies. The return on \$1 investment in strategy A is $(1+.07)^2$, whereas the return on investment B is $(1+.05) \times (1+ {}_1F_1)$, where ${}_1F_1$ is the forward rate for a one-year zero-coupon bond, one year from now. For the investor to be indifferent, the return on these two investments should be equal. Using algebra, one can solve for ${}_1F_1$:

Equation 4-1
Solving for ${}_1F_1$, One-Year Forward Rate

$$(1+.07)^2 = (1+.05) \times (1+ {}_1F_1)$$
$${}_1F_1 = 0.0904$$
$${}_1F_1 = 9.04\%$$

This methodology can be used to find forward rates for any maturity for any year in the future, given the appropriate rates. For example, if a zero-coupon bond with a maturity of three years yields 7.5%, we can then use the above yields to find the forward rate for a two-year bond, one year hence, and a one-year bond, two years hence, as shown below:

Finding the forward rate for a two-year bond, one year forward:

$$(1+.075)^3 = (1+.05) \times (1+{}_1F_2)^2$$
$${}_1F_2 = 0.0877$$
$${}_1F_2 = 8.77\%$$

Finding the forward rate for a one-year bond, two years forward:

$$(1+.075)^3 = (1+.07)^2 \times (1+{}_2F_1)$$
$${}_2F_1 = 0.0851$$
$${}_2F_1 = 8.51\%$$

Given an entire spot-rate yield curve that is the graph of interest rates as a function of time, one can find a forward yield curve for each year in the future.

■ STRIPPING THE PAR-YIELD CURVE

Notice that in the above discussion, the bonds were specified to be 'zero coupon.' This means that no coupon is paid out during the life of the bond. A lump sum of money is paid out on the maturity date. The absence of a coupon allows us to use the simple compounding formula in Equation 4-1 to derive the forward rates. If coupons are paid, then the computation is not as straightforward since one must take into account the reinvestment of the coupon payments. To avoid confusion, for the rest of this discussion, we will refer to the current yield curve associated with zero-coupon bonds as the spot curve and the individual zero-coupon yields as the spot rate. The common names for the different yield curves are shown in Table 4-1.

Table 4-1

Yield Curve Terminology

Spot Curve	The yield curve associated with bonds that do not pay any coupon during the life of the bond. Instead, zero-coupon bonds are traded at a discount (i.e. at a price less than face value). The spot curve rates can be taken from actual bonds being actively traded as well as artificially derived from stripping the par-yield curve.
Par-Yield Curve	The yield curve associated with bonds that pay coupon periodically and that are priced at par (i.e. at 100% of the face value).
Forward Curve	The implicit curves that show the market's assessment of how it will value future cashflows. There is a forward yield curve for every period in the future. These curves can be derived from the spot and the par-yield curves. The forward curves can be expressed as either par-yields or spot rates.

In order to find forward rates associated with coupon paying bonds of different maturities, trading at par (termed the par-yield curve), one can use the following methodology:

1. Strip the par-yield curve to get the spot curve.
2. Use the spot curve to get the forward-spot curve.
3. Reconstitute the forward-spot curve to get the forward par-yield curve.

The "bootstrapping" method is used to strip the par-yield curve. Each coupon-bearing bond, if discounted at the appropriate spot rate, will be valued at par, or 100. That is the definition of the par-yield curve. Thus, if we know the first spot rate and the par-yield curve, we can solve for all the spot rates.

To derive the spot curve, we start with the bond whose maturity equals one period. The yield of this bond will be the first one-period spot rate.

The next spot rate can be found using the two-period par bond. Discount the first set of cashflows with the one-period spot rate. The second set of cashflows should be discounted by the two-period spot rate, which is unknown. However, we can solve for the two-period spot rate because by definition, the sum of the discounted cashflows equals 100. By extension, the third-period spot rate may be found using the third-period par bond, and so on.

For example, assume the par yield curve in Table 4-2 where each bond pays out an annual coupon:

Table 4-2
A Sample Par-Yield Curve

Year	1	2	3	4
Yield	5.0	6.5	8.0	9.0

The first spot rate (S_1) will simply be 5.0% because the interest will only be paid at the maturity date of one year. The next bond pays 6.5% interest every year for two years. The first year's cashflow, given a bond with a principal amount of \$100, will be \$6.50. In the second year, the cashflow includes both the \$6.50 interest payment plus the principal of \$100. To find the next spot rate, S_2, solve the following equation for S_2, where rates are in decimal (i.e. 0.05).

$$100 = \frac{CF_1}{(1+S_1)} + \frac{CF_2}{(1+S_2)^2}$$

$$100 = \frac{6.5}{(1+.05)} + \frac{106.5}{(1+S_2)^2}$$

$$(1+S_2)^2 = \frac{106.5}{100 - \frac{6.5}{(1+.05)}}$$

$$S_2 = \sqrt{\frac{106.5}{100 - \frac{6.5}{(1+.05)}}} - 1$$

$$S_2 = 0.06549$$

$$S_2 = 6.549\%$$

To reconstitute a spot curve into a par-yield curve, just do the above calculation in reverse. That is, instead of solving for the spot rates, solve for the interest cashflow that, when discounted at the spot rates, will equal par. Equation 4-2 shows the generic equation for either stripping the par-yield curve or reconstituting the spot curve. It is simply the price-yield equation described in Chapter 3 (Equation 3-11), with the price set equal to par.

$$100=\frac{CF_1}{(1+S_1)^1}+\frac{CF_2}{(1+S_2)^2}+\frac{CF_3}{(1+S_3)^3}+\ldots\frac{CF_n}{(1+S_n)^n}$$

Equation 4-2
Solve for S_n to Strip or CF_n to Reconstitute the Curve

Exercise 4-1

Given the par coupon yield curve below, strip the curve to find the spot curve, then calculate all the implied spot forward rates.

Year	1	2	3	4
Yield	7.0	8.0	9.0	10.0

Hint: You can calculate six forward rates.

Work area

Year of Maturity	1	2	3	4
Spot Curve				

	Year Forward		
Maturity	1	2	3
1			
2			
3			

Exercise 4-2

Determine the forward rates from the following spot curve. Then determine the par-yield curve from the spot rates and the forward par-yield curve from the forward spot rates.

Year	1	2	3	4
Yield	10.0	9.0	8.0	7.0

Work area

Maturity	1	2	3	4
Spot Curve	10.0	9.00	8.00	7.00
Par Curve				

Forward spot rates:

	Year Forward		
Maturity	1	2	3
1			
2			
3			

Forward par rates:

	Year Forward		
Maturity	1	2	3
1			
2			
3			

VOLATILITY OF INTEREST RATES

Because the future is unknown, MBS valuation requires some kind of projection of future interest rates. While the forward rates described in the previous section are the market's prediction of future interest rates, we would like to create a set of possible interest rate scenarios. In order to create interest rate scenarios, one must have some idea of what type of movements are plausible. Interest rate volatility is one measure to describe the way interest rates change. The volatility of interest rates refers to the standard deviation of yield movements. A standard deviation measures the extent of variation around the mean. Interest rate volatility is normally stated in terms of an annualized percentage. Yield volatility is an important input into option valuation models.

The Mean

To calculate the extent of variation around the mean, one must first find the sample mean, $\bar{x}$, or average of a set of n measurements, x_1, $x_2, \ldots, x_n$. This is expressed as:

Equation 4-3
The Sample Mean

$$\bar{x} = \frac{\sum_{i=1}^{n} x_i}{n}$$

The Variance and Standard Deviation

The variation of data points around the mean is reflected in the difference of each data point from the mean: $(x_i - \bar{x})$. These are called deviations. To find an overall indicator of deviation for the entire set of data points, we might want to take the average of these deviations. The average of the deviations, however, would equal zero. In order to obtain a non zero number, we can calculate the sample variance, s^2, which is an average of the squared deviations:

$$s^2 = \frac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{n-1}$$

This can be calculated more easily as:

$$s^2 = \frac{\sum_{i=1}^{n} x_i^2 - n\bar{x}^2}{n-1}$$

Equation 4-4
The Sample Variance

The standard deviation, s, is simply the square root of the variance.

Calculating Yield Volatility

Given daily yield data, it is a simple exercise to calculate yield volatility for a particular maturity. The procedure is as follows given we have daily yields for a particular security, Y_i, where $i = 1, 2, \ldots, n$:

1. Calculate the relative yield change, Y_i/Y_{i-1} for each day.
2. Take the natural log of the daily relative change, $\ln(Y_i/Y_{i-1})$ for each day.
3. Calculate the standard deviation of the daily log relative changes.
4. The standard deviation of the daily log relative change is converted to an annual percentage basis by scaling the measure by the number of trading days in a year. This is usually approximated to be 250 (365 minus weekends and some holidays). To convert the daily standard deviation to the annual standard deviation, multiply by the square root of 250.

$$\text{Annual Standard Deviation} = \text{Daily Standard Deviation} \times \sqrt{\text{trading days per year}}$$

Equation 4-5
Finding Annual Standard Deviation

A two standard deviation range of interest rates represents a 95% confidence interval for interest rates. Thus, plotting the lower and upper ranges of possible interest rates within this 95% confidence interval can give a sense of the range of possible interest rate fluctuation.

Exercise 4-3

Given the tables below, calculate the yield volatility, expressed as an annual rate for one-year Treasuries and 30-year Treasuries. Then find the standard deviation measure stated in basis points. Does volatility change with maturity?

Constant Maturity 1-Year Treasury Bill Yields

Date	Yield	Date	Yield
2/13/95	6.82	2/20/95	6.60
2/14/95	6.73	2/21/95	6.65
2/15/95	6.64	2/22/95	6.46
2/16/95	6.57	2/23/95	6.50
2/17/95	6.61	2/24/95	6.47

30-Year Treasury Bill Yields

Date	Yield	Date	Yield
2/13/95	7.67	2/20/95	7.59
2/14/95	7.61	2/21/95	7.61
2/15/95	7.56	2/22/95	7.54
2/16/95	7.57	2/23/95	7.55
2/17/95	7.59	2/24/95	7.53

Work area

One-Year Volatility:

Date	Yield	Y_i/Y_{i-1}	$\ln(Y_i/Y_{i-1})$
2/13/95			
2/14/95			
2/15/95			
2/16/95			
2/17/95			
2/20/95			
2/21/95			
2/22/95			
2/23/95			
2/24/95			
Average			
Variance			
Daily Stan Dev			
Annual			
SD in BP			

30-Year Volatility:

Date	Yield	Y_i/Y_{i-1}	$\ln(Y_i/Y_{i-1})$
2/13/95			
2/14/95			
2/15/95			
2/16/95			
2/17/95			
2/20/95			
2/21/95			
2/22/95			
2/23/95			
2/24/95			
Average			
Variance			
Daily Stan Dev			
Annual			
SD in BP			

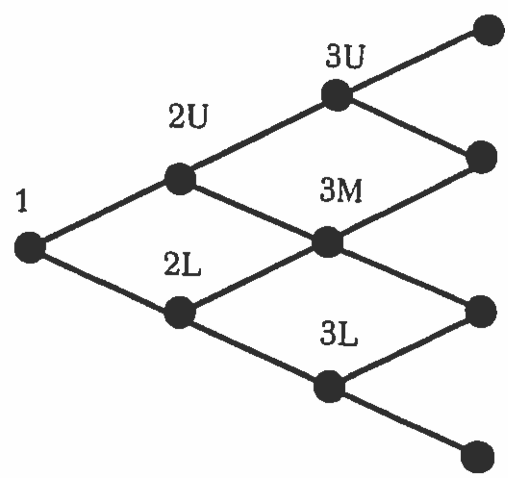
BINOMIAL TREES[1]

In the price-yield calculation from Chapter 3, interest rates are assumed to stay constant, which means mortgage cashflows are discounted with a single rate. In real life, interest rates are not constant over time. When cashflows are certain, discounting cashflows with today's yield curve is today's best valuation. But mortgage cashflows are not certain. In fact, they are highly dependent on interest rates through the prepayment option. Thus, one would like to know what will happen to our cashflows when interest rates change. There are many methods for modeling future interest rates. Here we discuss the binomial tree. We will build a tree of interest rates based upon the current yield curve and then use this tree to value bonds and options.

For bonds with embedded interest-rate dependent options, binomial models are the basic form of analysis. However for MBS, additional features are needed. These additional features will be discussed in Chapter 9. The models presented here are extremely simplified. In practice, there are a rich set of approaches to value these types of options.

A binomial tree takes the form shown in Figure 4-1. The first node, called the root, is simply the first period yield. The second level of nodes represents probable yields one period from now, while the third level of nodes represents probable rates two periods from now.

Figure 4-1
Binomial Tree



[1]All the binomial trees in this chapter were computer-generated using the *Yield Curve Primer* software, Andrew Kalotay Associates.

Each vertical set of nodes represents the rate possibilities for a time period in the future. A binomial tree using a discrete version of the log-normal process is characterized by three parameters:

1. A yield curve.
2. The relative probability of up and down moves.
3. Volatility.

We are assuming that the volatility is constant over all periods and that at each node, the probability of going up or down is equal. Thus, we will assume that the ratio between the upper and lower numbers in each of the nodes is e^{2V} where V is the short-term volatility in percentage per year, and e equals 2.718. The binomial tree must be consistent with the forward curve for no arbitrage. (This is a simplified explanation of the assumptions that go into a binomial tree. A complete treatment is beyond the scope of this chapter.)

The reasoning is as follows: at each node, one must determine the numbers such that (1) the proper ratio as described above holds true and (2) when a bond is valued using the binomial tree, the price will be equal to the current market price. Thus, an up branch to the right of any node represents an increase in yield, while a down branch is a decrease in yield, subject to the above two constraints.

Calculating a Sample Binomial Tree

Calculating a binomial tree involves numerical procedures. Unfortunately, the choice is between a trial and error approach or rather tedious closed-form solutions. The process for calculating a binomial tree is as follows. Consider the following par-yield curve in Table 4-3:

Table 4-3

A Sample Par-Yield Curve

Maturity	1	2	3
Par Yield	5.0	6.0	6.5

Assume that the annual volatility is 20%. Thus, the ratio between two nodes of the same level is $e^{0.40}$, which equals 1.49. The root is simply 5% or 0.05, as shown in Figure 4-2. To calculate the second level nodes, you first make a guess. A two-year bond would currently yield 6%, given the par-yield curve. The upper node should be a little higher and the lower node should be a little lower than 6%. If you make a guess for the lower node, simply multiply $e^{0.40}$ by your guess to get the upper node. If the two-year bond's cashflows are discounted by the correct tree, then the present value of the bond should equal 100, since that is the market's current valuation for the bond. (Remember, by definition, the par-yield curve is composed of bonds priced at 100, for different maturities.)

Figure 4-2
The Root of the Binomial Tree

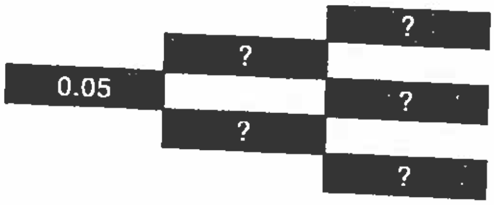


Figure 4-3 shows the guesses for the second node. They are, in fact, the correct rates to show how the calculation should work. To demonstrate, take the two-year bond, whose cashflows are shown in Table 4-4:

Figure 4-3
Guesses for the Second Node of the Binomial Tree

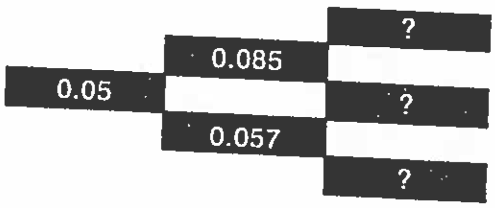


Table 4-4
Cashflows of Two-Year 6% Bond Priced at Par

Year	Cashflow
1	$6.00
2	$106.00

Table 4-5
Present Value of Two-Year 6% Bond from the Binomial Tree

Period	Lower	Upper
1	101.232	98.768
2	100.294	97.706

Steps to Calculate the Second Level[2]

1. Discount the $106 by your guesses for the upper and lower nodes. 8.5% and 5.7%, respectively. (Notice that 5.7% × 1.49= 8.5%. This is shown in Figure 4-3 for the second period.) The discounted values shown in period two of Table 4-5 represent what the two-year bond would be worth one year from now.

2. If you add the first period cashflow, $6, to the upper and lower values, you get $106.294 and $103.706.

3. Then, discounting these values by 5%, one gets the first period values shown in Table 4-5.

4. If you average these two values together, you get $100: ((101.232 + 98.768)/2)=100. Alternatively, one could have averaged in Step 2 and then discounted the average by 5%. If one did not get $100 at this step, that means that the guess was incorrect. If the answer was greater than 100, the guess was too low. If your answer was lower than 100, the guess was too high.

Steps to Calculate the Third Level

1. Taking the cashflows of a three-year bond at 6.5%, discount the third-year cashflow with your three guesses shown in Figure 4-4. The cashflows of the bond are shown in Table 4-6. The discounted values, shown in Table 4-7 for Period 3, represent what the bond would be worth two years from now, given the different interest rate scenarios.

Figure 4-4
The Guesses for the Third Node

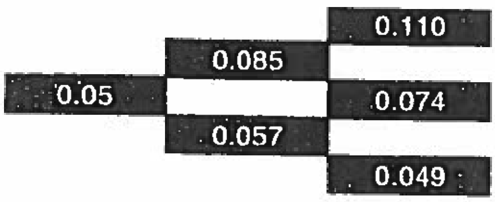


Table 4-6
Cashflows of Three-Year 6.5% Bond

Year	Cashflow
1	$6.50
2	$6.50
3	$106.50

[2]Many of the values in this and subsequent chapters were calculated in a spreadsheet, which carries out numbers to further decimal places than shown in the text. Thus, manual calculations will be slightly off.

Table 4-7

Present Value of Three-Year 6.5% Bond from the Binomial Tree

Period	Lower	Middle	Upper
1		105.000	
2	101.080		95.920
3	101.481	99.182	95.940

2. Add the second period cashflow, $6.50, to the values to obtain $107.981, $105.682, and $102.440. Average the upper and middle values, and the middle and lower values.
3. Discount these two values by the appropriate second node tree yields, to obtain the values shown in period two of Table 4-7.
4. Again, add the first period cashflows to these values. Average the two values, and discount by 5%.

$$\frac{(101.08+6.50)+(95.92+6.50)}{2}=105$$

$$\frac{105}{(1+.05)^1}=100$$

This process can be repeated to create a tree for the entire yield curve.

Jensen's Inequality

Intuitively, we would expect that the average rate for any period in a binomial tree would be equal to that period's forward rate. Compare the average rates for different volatility assumptions in Table 4-8. The third column shows the forward rates. The fourth column shows the average rates for a tree with zero volatility, and the fifth column shows the average rates given 20% volatility. Notice that the fifth column is not equal to the third and fourth columns. Our intuition is incorrect. Why?

Table 4-8

Average Rates for a Binomial Tree

Period	Par Yields	Forward Rates	0% Volatility	20% Volatility
1	6.000	6.000	6.000	6.000
2	9.139	12.371	12.371	12.425
3	11.355	15.924	15.924	16.169
4	13.787	21.406	21.406	22.204

The answer lies in Jensen's Inequality. Jensen's Inequality states that, in general, for an positive random variable $\tilde{X}$,

$$E\left(\frac{1}{\tilde{X}}\right) > \frac{1}{E(\tilde{X})}$$

Equation 4-6
Jensen's Inequality

where E stands for expected value.

When a binomial tree is created using the method outlined in the previous section, the tree's rates are found given today's bond prices. Thus, the tree rates represent an average price, not an average of the rates. This methodology correctly averages different bond prices, as represented on the left side of Equation 4-6. If we, instead, created the binomial tree using the forward rates as the middle path, and then used a volatility assumption to find the branches, we would undervalue the price of the bond as indicated by the inequality. When using bond pricing methodologies other than the binomial tree method, such as Monte Carlo simulation, one must correct for Jensen's Inequality.

Exercise 4-4

Calculate 1/average of X and the average of 1/X where X ={1,5,9}. Repeat this exercise using X ={4,5,6}.

	Set 1	Set 2
avg (x)		
1/av(x)		
avg(1/x)		

Valuing a Bond from a Binomial Tree

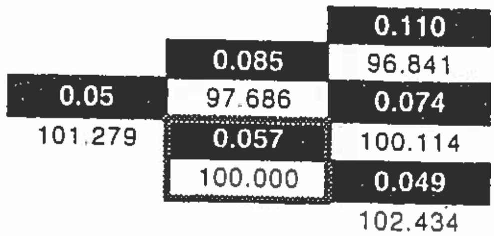
A bond can be valued using a binomial tree by following the same process as when creating the tree, the only difference being that no guessing of interest rates is necessary. A binomial tree is very useful to calculate the value of bonds with options.

A Bond with a Call Option

Consider Figure 4-5. The binomial tree is the one calculated in Figure 4-4. The numbers below each yield number correspond to the value of a three-year bond with a 7.5% coupon. The price of the bond today is 102.668. Now, say the bond is callable at the end of year 1 at par. If the bond's value in year 1 is greater than 100, the bond will be called. In Figure 4-6, the outlined node corresponds to where the bond is called. The present value of the bond at the root of the tree has changed from 102.668 to 101.279. This change in value represents the option price, in this case 1.389.

Figure 4-5
Valuation of a Three-Year 7.5% Optionless Bond Using a Binomial Tree

Figure 4-6
Bond Callable at Par at End of First Year



A Bond with a Put Option

Consider Figure 4-7. This tree values the same bond, but with a put option at the end of year 1, instead of a call option. If the price of the bond is less than 100, the option will be exercised. Again, the price change reflects the price of the option, 1.102.

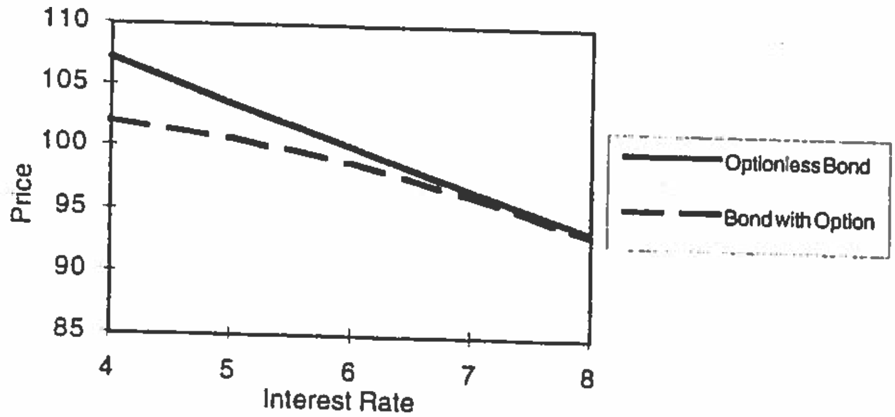
Figure 4-7
Bond Putable at Par at End of First Year

The embedded call options in MBS have a very significant effect on price. An optionless bond will have a positively *convex* price profile

Figure 4-8 shows the price profile of a 6% bond with and without a call option. The call option can be exercised at par at the end of every period. As interest rates go down, the price of both of the bonds goes up. However, the bond with the call option does not increase in price as rapidly as does the optionless bond. This *concave* curvature for the bond with the call option is termed "negative convexity," which will be described in more detail in Chapter 7.

Figure 4-8

Price Profile of Bond with and without Options



The binomial tree is one method of implementing a single factor term-structure of interest rates. This method offers easy implementation but neglects the role of the yield curve shape on MBS cashflows. Another technical problem is the disregard for path dependency. Multiple factor models using Monte Carlo simulation techniques offer a richer simulation of future yields but at the cost of increased computation time.

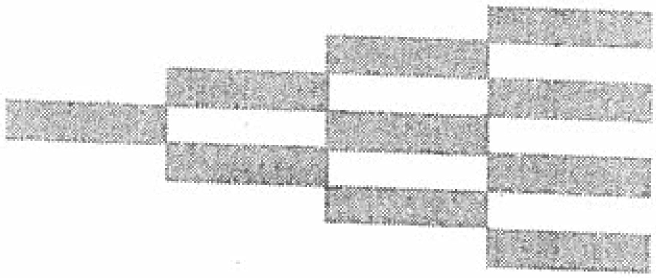
Exercise 4-5

Given the following par yield curve, calculate a binomial tree for 15% volatility, and then for 0% volatility.

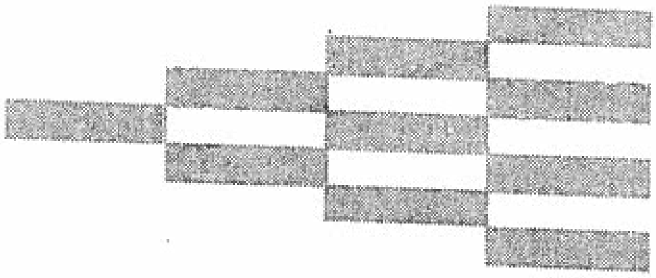
Period	1	2	3	4
Yield	5.0	5.5	6.5	7.5

Work area

15% Volatility



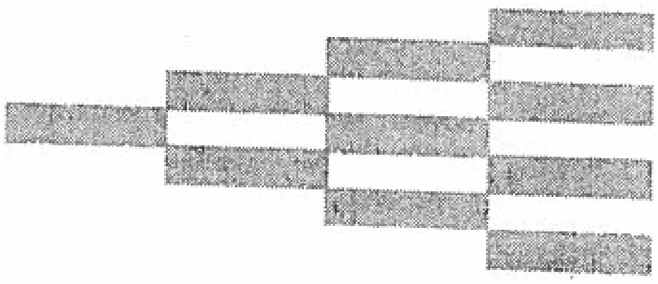
0% Volatility



Exercise 4-6

Price a four-year bond with an 8% coupon off of the binomial tree with 15% volatility.

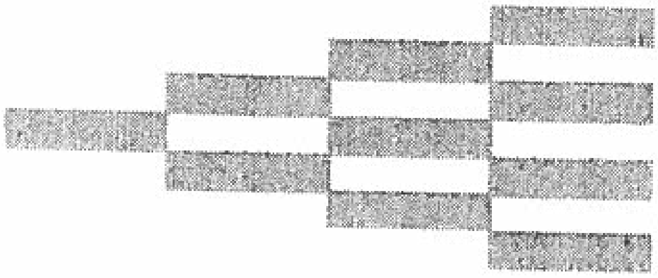
Coupon 8%
15% Volatility



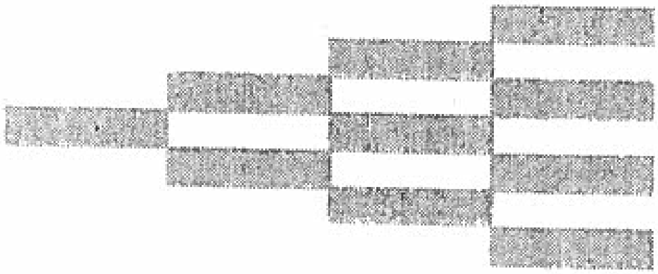
Exercise 4-7

If the bond in Exercise 4-6 has a call option at the end of the third year at par, what is the value of the option? What if it has a put option in the third year at 98?

Bond callable at the end of the third year at par
Option Value:



Bond putable at the end of the third year at 98
Option Value:



■ REVIEW QUESTIONS

If the par yield curve is upward sloping, what does that say about spot rates and one period forward rates? Are spot rates higher or lower than par rates? Are forward rates higher or lower than par rates?

If the 30-year zero-coupon bond (strip) has a higher yield than the 30-year coupon bond, is the strip a better investment?

What effect does the volatility assumption have on option value?

Prepayments are linked to a variety of economic factors. How would you incorporate a factor such as GNP growth into a binomial model?

ANSWERS TO EXERCISES

4-1

Year of Maturity	1	2	3	4
Spot Curve	7.00	8.04	9.13	10.27

	Year Forward		
Maturity	1	2	3
1	9.09	11.33	13.79
2	10.20	12.55	
3	11.38		

4-2

Maturity	1	2	3	4
Spot Curve	10.0	9.00	8.00	7.00
Par Curve	10.0	9.04	8.10	7.17

Forward zero rates:

Maturity	Year Forward 1	2	3
1	8.01	6.03	4.06
2	7.01	5.04	
3	6.02		

Forward par rates:

Maturity	Year Forward 1	2	3
1	8.01	6.03	4.06
2	7.05	5.06	
3	6.10		

4-3

1-Year Volatility:

Date	Yield	Y_i/Y_{i-1}	$\ln(Y_i/Y_{i-1})$
2/13/95	6.82		
2/14/95	6.73	0.987	–0.013
2/15/95	6.64	0.987	–0.013
2/16/95	6.57	0.989	–0.011
2/17/95	6.61	1.006	0.006
2/20/95	6.60	0.998	–0.002
2/21/95	6.65	1.008	0.008
2/22/95	6.46	0.971	–0.029
2/23/95	6.50	1.006	0.006
2/24/95	6.47	0.995	–0.005
Average	6.605		–0.006
Variance			0.000
Daily Stan Dev			0.012
Annual			19.02%
SD in BP			125.62

30-Year Volatility:

Date	Yield	Y_i/Y_{i-1}	$\ln(Y_i/Y_{i-1})$
2/13/95	7.67		
2/14/95	7.61	0.992	–0.008
2/15/95	7.56	0.993	–0.007
2/16/95	7.57	1.001	0.001
2/17/95	7.59	1.003	0.003
2/20/95	7.59	1.000	0.000
2/21/95	7.61	1.003	0.003
2/22/95	7.54	0.991	–0.009
2/23/95	7.55	1.001	0.001
2/24/95	7.53	0.997	–0.003
Average	**7.582**		**–0.002**
Variance			**0.000**
Daily Stan Dev			**0.005**
Annual			**7.44%**
SD in BP			**56.43**

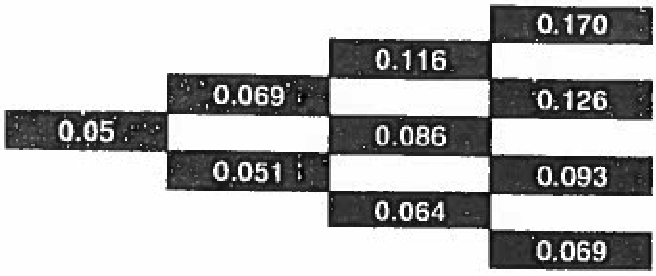
4-4

	Set 1	Set 2
avg (x)	5.000	5.000
1/av(x)	0.200	0.200
avg(1/x)	0.437	0.206

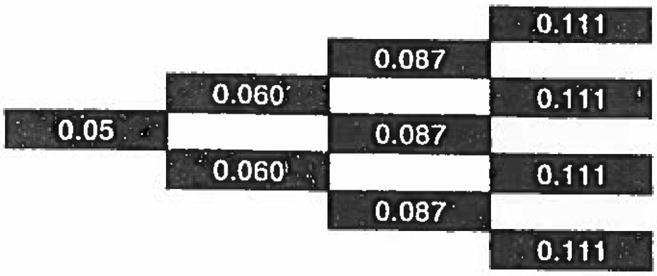
Notice that the inequality is greater as the volatility of x increases.

4-5

15% volatility

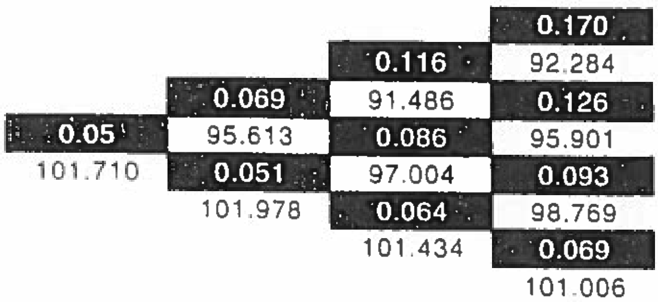


0% Volatility
Note that these are simply the forward rates.



4-6

Coupon 8%
15% Volatility



4-7

Bond callable at the end of the third year at par
Option Value: 0.107

			0.170
		0.116	92.284
	0.069	91.486	0.126
0.05	95.613	0.086	95.901
101.603	0.051	97.004	0.093
	101.753	0.064	98.769
		100.961	0.069
			100

Bond putable at the end of the third year at 98
Option Value: 1.214

			0.170
		0.116	98.000
	0.069	94.988	0.126
0.05	97.702	0.086	98.000
102.924	0.051	97.970	0.093
	102.438	0.064	98.769
		101.434	0.069
			101.006